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Sujet de la thèse :

Étude de certains phénomènes en milieux aléatoires

Soutenue le 21 novembre 2005, devant le jury composé de :

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A mes parents

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Chapitre 1

Introduction

Nous nous intéressons dans ce qui suit à trois modèles en milieux aléatoires en dimension un :

- les marches aléatoires en milieu aléatoire,
- les diffusions dans un potentiel brownien avec ou sans drift,
- un système de marches aléatoires en milieu aléatoire avec branchement.

L'étude de ces modèles nécessite des approches variées. On regarde plus particulièrement des propriétés liées aux grandes déviations et au maximum du temps local.

Nous présentons les marches aléatoires en milieu aléatoire dans la première partie. La deuxième partie contient un rappel des propriétés importantes de ce modèle, ainsi que des diffusions dans un potentiel aléatoire. Les principaux résultats de cette thèse sont présentés dans la troisième partie.

1 Présentation des marches aléatoires en milieu aléatoire

1.1 Motivation

Les marches aléatoires usuelles permettent de modéliser de nombreux phénomènes : ruine des joueurs, approximation du mouvement brownien, étude des gaz, etc. Cependant, dans ce modèle l'espace est homogène, et ne prend pas en compte les imperfections de l'environnement, par exemple des impuretés dans un métal, ou la présence d'un champ de forces aléatoire. Or, dans de nombreux systèmes physiques et biologiques, ces défauts de l'environnement sont loin d'être négligeables, et donnent naissance à des phénomènes inattendus. Il est donc apparu nécessaire d'introduire de nouveaux modèles qui prennent en compte l'hétérogénéité spatiale résultant de ces défauts.

Différentes approches ont émergé. Nous nous intéressons ici plus spécifiquement aux marches aléatoires en milieu aléatoire, introduites en 1967 par le biophysicien Chernov [14]. Nous les notons dans la suite RWRE pour « random walks in random environment ». En dimension un, les RWRE sont définies de manière informelle de la façon suivante :

– La première étape consiste à choisir, aléatoirement, un coefficient $\omega_i \in]0,1[$ pour tout entier $i \in \mathbb{Z}$. Ces coefficients constituent l'*environnement*, ou le *milieu*, qui est donc bien aléatoire. Cet environnement peut être représenté par la figure 1.1.

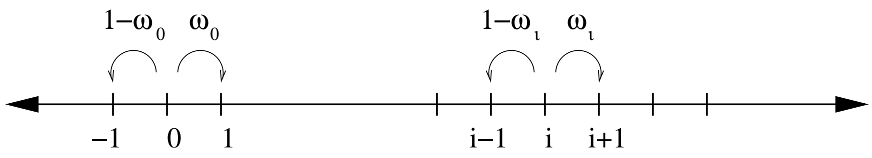

Fig. 1.1 – Milieu aléatoire en dimension 1

– Dans une deuxième étape, on considère une particule, située à la position 0 à l'instant 0. Avec probabilité ω_0, elle saute à la position 1 à l'instant 1, sinon elle saute à la position -1. De même, si elle est au point i à l'instant n, elle va en $i+1$ à l'instant $n+1$ avec probabilité ω_i, sinon elle va en $i-1$, et ainsi de suite.

Les marches aléatoires en milieu aléatoire en dimension supérieure sont définies de façon similaire. Nous verrons par la suite que les RWRE présentent des comportements très différents de ceux des marches aléatoires usuelles, à cause de la compétition entre deux couches d'aléa, l'environnement d'une part, et le mouvement de la particule d'autre part.

Les RWRE sont utilisées dans des contextes variés. Tout d'abord, Chernov [14] les introduit dans l'objectif de modéliser la réplication de l'ADN. De plus, les généticiens recourent actuellement aux RWRE dans d'autres situations. Par exemple, Lubensky et Nelson [58] se servent de ce type de modèle pour étudier la micro manipulation des brins d'ADN.

Temkin [82] reprend quant à lui ce modèle en 1972 avec des motivations liées non seulement à la génétique mais aussi à la métallurgie, et plus particulièrement à la cinétique des transitions de phase dans des alliages.

Les RWRE sont aujourd'hui étudiées à la fois par des physiciens et des mathématiciens, et s'appliquent à l'étude du comportement de nombreux phénomènes de diffusion et de transport dans des milieux spatialement hétérogènes.

1.2 Définition en dimension un

Les marches aléatoires en milieu aléatoire à une dimension sont définies de la façon suivante.

Définition 1.1 *Soit $(\omega_i)_{i\in\mathbb{Z}}$ une suite de variables aléatoires indépendantes et identiquement distribuées. Une réalisation de ces variables aléatoires est appelée un environnement. Étant donné un environnement $\omega = (\omega_i)_{i\in\mathbb{Z}}$, on appelle marche aléatoire en milieu aléatoire la chaîne de Markov $(S_n)_{n\in\mathbb{N}}$ définie par $S_0 = 0$ et pour $n \geq 0$,*

$$P_\omega(S_{n+1} = k | S_n = i) = \begin{cases} \omega_i & si\ k = i+1, \\ 1-\omega_i & si\ k = i-1, \\ 0 & sinon. \end{cases} \tag{1.1}$$

Remarquons que lorsque la loi de ω_0 est une masse de dirac, $(S_n)_{n\in\mathbb{N}}$ est une marche aléatoire usuelle ; nous évitons ce cas dégénéré dans la suite. Nous verrons que le comportement de $(S_n)_{n\in\mathbb{N}}$ dépend considérablement de la loi de l'environnement ω, notée η.

Il convient de signaler que l'on distingue deux lois de probabilités différentes. La loi de probabilité conditionnellement à l'environnement, notée P_ω et définie ci-dessus, est appelée la loi quenched (c'est à dire « trempé », selon la terminologie provenant de la métallurgie). Cette loi présente l'avantage d'être markovienne, mais elle ne possède pas de propriété d'invariance par translation. La loi annealed (« recuite ») est quant à elle définie comme la moyenne de la loi quenched sur tous les environnements, c'est à dire $\mathbb{P}(.) = \int P_\omega(.)\eta(\mathrm{d}\omega)$, où η est la loi de l'environnement. Il s'agit d'une loi invariante par translation mais non markovienne. Certains phénomènes, comme les grandes déviations notamment, ont des propriétés différentes sous ces deux lois.

D'autres modèles similaires sont étudiés, avec des techniques très différentes. Par exemple, Key [54], Derriennic [26] et Brémont [9] autorisent la marche à faire des sauts de taille variable. Voir également Kirkpatrick [55] pour un modèle de conductivité aléatoire, de Gennes [34], Berger, Gantert et Peres [5] au sujet de marches aléatoires sur des amas de percolation, Dembo, Gantert, Peres, Zeitouni [22] pour des marches aléatoires sur des arbres aléatoires, et plus généralement Bolthausen et Sznitman [8] pour divers modèles en milieux aléatoires.

L'étude des RWRE en dimension supérieure s'avère particulièrement difficile. Les méthodes employées sont très différentes de celles employées en dimension 1. Citons néanmoins quelques références pour donner un aperçu du sujet. Kalikow [45] établit un critère de transience, Sznitman et Zerner [77], puis Comets et Zeitouni [19] et Enriquez et Sabot [27] prouvent des lois des grands nombres, Sznitman [75] un théorème central limite, et Varadhan [83] étudie les grandes déviations. Voir également Adelman et Enriquez [1] sur les informations données par une trajectoire sur la loi de l'environnement, les résultats récents de Sabot [67], Fontes et Mathieu [29], Mathieu et Piatniski [62] et plus généralement les cours de Sznitman [76] et Zeitouni [84].

Nous verrons par la suite que les RWRE et les marches aléatoires usuelles ont des propriétés très différentes. L'objectif de la partie suivante est de donner un aperçu de la richesse des comportements en milieu aléatoire (en dimension un), richesse qui résulte de la compétition entre l'environnement et l'agitation thermique.

2 Un rapide tour d'horizon des résultats connus en environnement aléatoire

Dans la partie 2.1, nous présentons les résultats les plus classiques concernant les RWRE. Puis, dans la partie 2.2, nous introduisons un modèle continu analogue aux RWRE : les diffusions dans un potentiel aléatoire. Nous évoquons ensuite les résultats concernant les grandes déviations pour les RWRE et les diffusions dans un potentiel brownien avec ou sans drift dans la partie 2.3. Enfin, nous nous intéressons au maximum du temps local pour ces deux processus dans la partie 2.4.

2.1 Premières propriétés des marches aléatoires en milieu aléatoire

On se place dans toute la suite sous l'hypothèse suivante

$$\exists \varepsilon_0 > 0, \qquad \eta(\varepsilon_0 < \omega_0 < 1 - \varepsilon_0) = 1. \tag{2.1}$$

On définit par ailleurs

$$\rho_i := \frac{1 - \omega_i}{\omega_i}.$$

Pour éviter le cas dégénéré des marches aléatoires usuelles, on suppose dorénavant que

$$\sigma^2 := \mathbb{V}ar(\log \rho_0) \neq 0.$$

Les critères de récurrence et de transience des RWRE sont déterminés en 1974 par Solomon [74]. Il prouve que la marche $(S_n)_{n\in\mathbb{N}}$ est récurrente si et seulement si $\mathbb{E}(\log \rho_0) = 0$. Il établit de plus une loi des grands nombres : il existe une vitesse limite $v \in [-1, 1]$, ne dépendant que de la loi de l'environnement, telle que

$$\frac{S_n}{n} \underset{n\to+\infty}{\longrightarrow} v \qquad \mathbb{P}\text{–p.s.}$$

Contrairement aux marches aléatoires usuelles, les RWRE peuvent avoir une vitesse v nulle même dans le cas transient. En 1975, Kesten, Kozlov et Spitzer [53] donnent dans le cas transient un théorème de type central limite dans lequel apparaissent trois régimes différents, en fonction du paramètre s, unique solution de l'équation $\mathbb{E}(\rho_0^s) = 1$.

Sinai [72] montre en 1982 que dans le cas récurrent, les RWRE sont beaucoup plus lentes que les marches aléatoires usuelles. Plus précisément, il existe une variable aléatoire non dégénérée et non gaussienne b_∞, telle que

$$\sigma^2 \frac{S_n}{(\log n)^2} \xrightarrow[n\to+\infty]{loi} b_\infty. \tag{2.2}$$

Ce phénomène contraste avec le cas des marches aléatoires usuelles qui ont un comportement asymptotique en $\sqrt{n}$. La démonstration de Sinai fait apparaître une quantité qui joue un rôle important, le potentiel V. On le définit comme suit pour $x \in \mathbb{Z}$:

$$V(x) := \begin{cases} \sum_{i=1}^{x} \log \rho_i = \sum_{i=1}^{x} \log \frac{1-\omega_i}{\omega_i} & \text{si } x > 0, \\ 0 & \text{si } x = 0, \\ -\sum_{i=0}^{x-1} \log \rho_{-i} = -\sum_{i=0}^{x-1} \log \frac{1-\omega_{-i}}{\omega_{-i}} & \text{si } x < 0. \end{cases} \tag{2.3}$$

Cette quantité dépend uniquement de l'environnement et joue un rôle analogue à une énergie en physique. Le caractère sous-diffusif (2.2) des RWRE établi par Sinai est dû à la

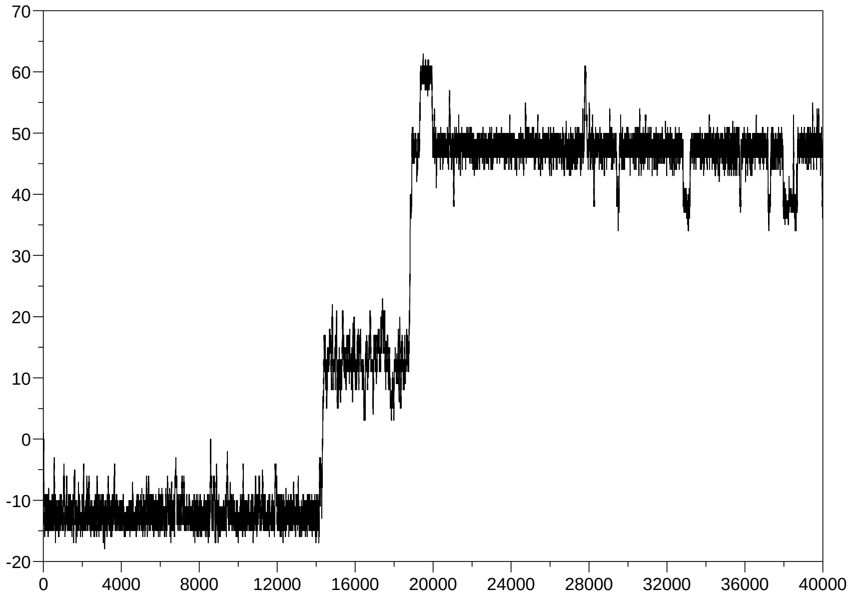

FIG. 1.2 – Marche aléatoire en milieu aléatoire récurrente, en dimension 1

présence de « trappes ». Ce sont des puits de potentiel (il s'agit de fonds de vallées pour le potentiel V) dans lesquels la marche $(S_n)_{n\in\mathbb{N}}$ reste piégée longtemps avant de réussir à s'en échapper. Elle est ensuite piégée par un autre puits de potentiel, dans lequel elle passe encore plus de temps, et ainsi de suite. On peut observer un tel phénomène sur la figure 2.1, où l'on distingue nettement trois pièges dans lesquels la marche passe l'essentiel de son temps. Golosov [35] localise ensuite plus précisément la particule, toujours dans le cas récurrent. De tels phénomènes de localisation de $(S_n)_{n\in\mathbb{N}}$ seront utiles dans le chapitre 2, pour des environnements ω bien choisis.

2.2 Diffusion dans un potentiel aléatoire

Afin de pouvoir disposer de la puissance du calcul stochastique, Schumacher [68] a l'idée d'associer aux RWRE une diffusion en milieu aléatoire. Soit un processus aléatoire $(V(x),\ x \in \mathbb{R})$. On considère la diffusion $(X(t),\ t \geq 0)$, définie comme la solution formelle de l'équation différentielle stochastique suivante

$$\begin{cases} \mathrm{d}X(t) = \mathrm{d}\beta(t) - \frac{1}{2}V'(X(t))\mathrm{d}t, \\ X(0) = 0, \end{cases} \tag{2.4}$$

où $(\beta(t),\ t \geq 0)$ est un mouvement brownien indépendant de V. Le processus X est appelé diffusion dans le potentiel aléatoire V. Si V n'est pas dérivable, l'équation précédente n'a qu'un sens formel. On définit alors plus rigoureusement X comme un processus de Markov dont le générateur infinitésimal, conditionnellement à V, est donné par la formule

$$\frac{1}{2}e^{V(x)}\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{-V(x)}\frac{\mathrm{d}}{\mathrm{d}x}\right).$$

En utilisant un argument de type Donsker, on constate que le potentiel d'une RWRE défini dans l'équation (2.3) est asymptotiquement un mouvement brownien (avec drift dans le cas transient). C'est pourquoi l'on étudie plus particulièrement la diffusion X dans un potentiel

$$W_\kappa(x) = W(x) - \frac{\kappa}{2}x,$$

avec $\kappa \in \mathbb{R}$, où W est un mouvement brownien indépendant de β. Le comportement des diffusions dans ce type de potentiel présente de nombreuses similitudes avec celui des RWRE.

Le cas récurrent, qui correspond à $\kappa = 0$, est introduit par Brox [10]. Il prouve, en utilisant l'auto-similarité du mouvement brownien, que ce processus X a un comportement asymptotique en $(\log t)^2$, comme les RWRE récurrentes. De même, Kawazu, Tamura et Tanaka [47] et Hu [39] localisent précisément la diffusion X dans un puits du potentiel, donnant ainsi un résultat analogue à celui de Golosov.

Dans le cas récurrent, les preuves sont souvent basées sur la localisation de la particule. Or, dans le cas transient, c'est à dire $\kappa \neq 0$, nous ne disposons pas de résultat de localisation, et d'autres techniques sont utilisées. Une première méthode se base sur le lemme de Kotani, qui relie la transformée de Laplace des temps d'atteinte de X conditionnellement à l'environnement à la solution d'une équation différentielle stochastique. A partir de ce lemme et de la théorie spectrale de Krein, Kawazu et Tanaka [49] prouvent qu'il existe trois régimes différents, en fonction de la valeur de κ :

- si $0 < \kappa < 1$, la vitesse est nulle et les temps d'atteinte de X sont polynômiaux ;
- si $\kappa > 1$, le processus X a une vitesse limite non nulle ;
- dans le cas critique $\kappa = 1$, le temps d'atteinte de r par X est asymptotiquement de l'ordre de $r \log r$.

Remarquons une fois de plus l'analogie entre ces trois régimes et ceux observés pour les RWRE par Kesten, Kozlov et Spitzer [53].

Ce résultat est précisé par les travaux de Kawazu et Tanaka [50], Tanaka [81], puis Hu, Shi et Yor [44], qui quantifient tous les taux de convergence des temps d'atteinte. En particulier, Hu et al. [44] développent une méthode nouvelle basée sur la transformation de Lamperti (voir Revuz et Yor [66], chap. XI) et les processus de Jacobi, méthode que nous exploiterons dans le chapitre 4.

Des diffusions dans d'autres types de potentiel aléatoire sont également beaucoup étudiées. Comme le potentiel d'une RWRE peut être asymptotiquement un processus stable si l'on supprime l'hypothèse (2.1), il est intéressant d'étudier les diffusions dans un potentiel stable. Voir notamment à ce sujet les travaux de Kawazu, Tamura et Tanaka [47], Cheliotis [13] et Singh [73]), et ceux de Carmona [11] pour des potentiels de type processus de Lévy.

D'autre part, Mathieu [61] ajoute au potentiel W_κ une fonction du type $x \mapsto x^\alpha$. Il étudie le comportement asymptotique d'une diffusion dans ce nouveau potentiel aléatoire, qui n'est plus auto-similaire, en fonction de la valeur de α. Par ailleurs, Tanaka [79, 80] généralise les diffusions en potentiel aléatoire en dimension supérieure, et Mathieu [59, 60] obtient des comportements asymptotiques en $(\log t)^a$ en étudiant les vallées du potentiel.

2.3 Grandes déviations

L'étude des grandes déviations concerne le comportement asymptotique de la probabilité de certains événements rares, du type $\mathbb{P}(X_n/n \in A)$ ou $P_\omega(X_n/n \in A)$, lorsque ces probabilités tendent vers 0.

En 1994, Greven et Den Hollander [36] prouvent que les RWRE satisfont un principe de grandes déviations sous la loi quenched (conditionnellement à l'environnement). La fonction de taux pour la loi quenched a la particularité d'être déterministe. Grâce à une expression de cette fonction sous la forme d'une fraction continue à coefficients aléatoires, ils obtiennent certaines de ses propriétés qualitatives. Cependant nous ne disposons pas de son expression explicite dans le cas général, et de nombreuses questions concernant les propriétés de cette fonction de taux restent ouvertes.

Comets, Gantert et Zeitouni [15] démontrent un principe de grandes déviations sous la loi annealed, et établissent l'existence de grandes déviations fonctionnelles. En particulier, les fonctions de taux pour les grandes déviations quenched et annealed sont différentes lorsque la fonction de taux annealed est non nulle. Noter que ce phénomène n'est plus vrai pour les marches aléatoires sur des arbres aléatoires étudiées par Dembo, Gantert, Peres et Zeitouni [22], pour lesquelles les fonctions de taux quenched et annealed coincident. Un des phénomènes importants concernant les grandes déviations est le lien qui existe en général entre la dérivée seconde de la fonction de taux en 0 et la sous-diffusivité. Nous nous intéressons à de telles propriétés dans le chapitre 2.

Taleb [78] quant à elle prouve que les diffusions dans un potentiel brownien, avec ou sans drift, admettent un principe de grandes déviations sous les lois quenched et annealed, et donne pour les fonctions de taux des expressions plus explicites, à l'aide de fonctions de Bessel.

Nous préciserons certaines propriétés de ces fonctions de taux dans le chapitre 2. D'autre part, nous présenterons une application des résultats de grandes déviations à l'étude d'un système de marches aléatoires en milieu aléatoire avec branchement dans le chapitre 3.

2.4 Fluctuations du temps local

Une des quantités importantes pour une RWRE $(S_n)_{n\in\mathbb{N}}$ est le temps local, défini par

$$\xi(n,x) := \#\{0 \le k \le n,\ S_k = x\}, \qquad x \in \mathbb{Z},\ t \in \mathbb{N}.$$

Il s'agit donc du nombre de visites au point x de la marche jusqu'à l'instant n. Révész [64, 65], puis Hu et Shi [41] déterminent dans le cas récurrent les fluctuations de $\xi(n,x)$ pour x fixé et montrent que le temps local d'une RWRE peut prendre des valeurs beaucoup plus importantes que celui d'une marche aléatoire usuelle. Ceci est cohérent avec le résultat (2.2) de Sinai. En effet, la marche aléatoire en milieu aléatoire est plus lente qu'une marche aléatoire usuelle, il est donc vraisemblable qu'à certains instants elle ait passé plus de temps que cette dernière en un point donné.

Il est alors naturel de chercher à mieux comprendre les fluctuations non plus du temps local en un point donné, mais du maximum du temps local

$$\xi^*(n) := \max_{x\in\mathbb{Z}} \xi(n,x).$$

Afin de comprendre à quel point ξ^* peut prendre des grandes valeurs, on étudie le problème suivant. Pour une fonction f donnée, on essaie de savoir s'il existe une infinité d'instants n pour lesquels $\xi^*(n)$ est au moins aussi grand que $f(n)$, ou si, au contraire, ξ^* est majoré par f à partir d'un certain temps.

On constate que le maximum du temps local ξ^* exhibe des propriétés particulièrement surprenantes qui témoignent de la très forte concentration à certains instants de la RWRE autour de ses sites les plus visités. En effet, Révész [65] puis Shi [70] montrent que pour une RWRE récurrente,

$$\limsup_{n\to+\infty} \frac{\xi^*(n)}{n} = c_1,$$

où c_1 est une constante strictement positive. Il existe ainsi une infinité d'instants auxquels la RWRE a passé en un seul point une fraction de son temps supérieure à c_1. Gantert et Shi [31] montrent que ce résultat reste vrai pour une RWRE transiente à vitesse nulle. En revanche, ils établissent que $\limsup_{n\to+\infty} \xi^*(n)/n = 0$ pour une RWRE à vitesse non nulle. Ils caractérisent dans ce cas les classes de Lévy au sens « $\limsup$ », c'est à dire les fonctions croissantes f pour lesquelles

$$\limsup_{t\to+\infty} X(t)/f(t) = +\infty \qquad \mathbb{P}\text{–p.s.}$$

et les fonctions croissantes g pour lesquelles

$$\limsup_{t\to+\infty} X(t)/g(t) = 0 \qquad \mathbb{P}\text{–p.s.}$$

Les mêmes questions se posent naturellement pour les diffusions en milieu aléatoire, dont le temps local est défini par la formule

$$L_X(x,t) := \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\int_0^t 1_{X(s)\in[x,x+\varepsilon]}\mathrm{d}s, \qquad t\geq 0,\ x\in\mathbb{R}.$$

On s'intéresse comme précédemment au maximum du temps local

$$L_X^*(t) := \sup_{x\in\mathbb{R}} L_X(t,x).$$

Shi [70] étudie cette quantité pour une diffusion dans un potentiel brownien sans drift, et prouve que
$$\limsup_{t\to+\infty} L_X^*(t)/(t\log\log\log t) > 0.$$
La majoration correspondante n'est pas encore établie.

Signalons également les problèmes intéressants concernant les points les plus visités, c'est à dire les points x pour lesquels $\xi(n,x)=\xi^*(n)$. Voir par exemple Erdős et Révész [28] pour les marches aléatoires usuelles, Hu et Shi [42] et Shi et Zindy [71] dans le cas des RWRE.

3 Principaux résultats

Cette section présente brièvement les principaux résultats obtenus. Certains énoncés sont reproduits ici, mais d'autres ne sont donnés que dans le chapitre correspondant.

3.1 Chapitre 2 : Quelques propriétés des fonction de taux de grandes déviations quenched d'une RWRE

Dans ce chapitre, nous nous intéressons à deux conjectures présentées par Greven et den Hollander [36] en 1994, et rappelées dans den Hollander [37] en 2000. Elles concernent des propriétés de la fonction de taux de grandes déviations quenched d'une RWRE.

Tout d'abord, ils conjecturent dans ([36], open problem 2) que dans le cas récurrent, la fonction de taux quenched I_η^q d'une RWRE a une dérivée seconde infinie en 0. Ceci correspondrait au caractère sous-diffusif de la marche. Le chapitre 2 apporte une réponse positive à cette question.

Théorème 3.1 *(voir Theorem 1.1, page 31) Si $(S_n)_{n\in\mathbb{N}}$ est une marche aléatoire en milieu aléatoire récurrente, alors*
$$\lim_{\theta\to0^+}(I_\eta^q)''(\theta)=+\infty.$$

La méthode utilisée consiste à déterminer le comportement asymptotique des dérivées successives de la transformée de Laplace du temps d'atteinte du point 1 par $(S_n)_{n\in\mathbb{N}}$, c'est à dire de
$$\tau_1 := \inf\{n>0,\ S_n=1\}$$
(voir Proposition 1.2, page 31). Nous construisons pour cela des environnements qui contraignent la particule à s'éloigner longtemps de l'origine puis nous localisons la particule dans un profond puits de potentiel.

Nous nous intéressons ensuite à la question « open problem 3 » énoncée dans Greven et Den Hollander ([36]. Les auteurs conjecturent que l'on peut comparer dans le cas transient la fonction de taux quenched d'une RWRE avec la fonction de taux de grandes déviations d'une marche aléatoire usuelle de même vitesse limite. Plus précisément, ils suggèrent que, dans l'échelle logarithmique, la probabilité que S_n/n soit supérieur à la vitesse limite est plus grande pour une RWRE que pour une marche aléatoire usuelle de même vitesse limite. Nous nous sommes intéressés au problème correspondant dans le cas continu, pour lequel la fonction de taux est plus explicite. Nous donnons une réponse positive à cette comparaison pour la diffusion X dans le potentiel $W_\kappa(x) = W(x) - \frac{\kappa}{2}x$. On note v_κ la vitesse limite de X, c'est à dire l'unique réel tel que $\lim_{n\to+\infty} X(t)/t = v_\kappa$ $\mathbb{P}$–presque sûrement.

Théorème 3.2 *(voir Theorem 1.3, page 32) Si $\kappa > 1$, alors la fonction de taux de grandes déviations quenched de X, notée J_κ vérifie l'inégalité*

$$\forall x > v_\kappa, \qquad J_\kappa(x) < J^B_{v_\kappa}(x),$$

où $J^B_{v_\kappa}$ est la fonction de taux de grandes déviations du mouvement brownien avec drift $(B(t) + v_\kappa t,\ t \geq 0)$.

Le méthode utilisée est purement analytique. On prouve au passage une inégalité à notre connaissance nouvelle sur des quotients de fonctions de Bessel modifiées (voir Proposition 1.4, page 32).

3.2 Chapitre 3 : Comportement asymptotique d'un système de marches aléatoires en milieu aléatoire avec branchement

Ce chapitre présente une application des résultats de grandes déviations pour les RWRE. Nous y étudions un système de particules qui se reproduisent selon un processus de Galton–Watson supercritique, et se déplacent dans un milieu aléatoire (celui de la figure 1.1) dans le cas « transient vers la gauche », c'est à dire $\mathbb{E}(\log \rho_0) > 0$.

Le modèle est défini plus précisément de la façon suivante. Soit une suite de réels $(p_k)_{k\in\mathbb{N}}$, tels que $p_k \geq 0$ pour tout k, et $\sum_k p_k = 1$. Considérons un environnement ω fixé.
— A l'instant $n = 0$, il y a uniquement une particule, située en 0.
— A l'instant $n = 1$, la particule saute en 1 avec probabilité ω_0, et en -1 avec probabilité $1 - \omega_0$. Arrivée à sa nouvelle position, elle donne naissance à k nouvelles particules avec probabilité p_k, et meurt.
— A l'instant $n = 2$ chaque particule bouge indépendamment (vers un site voisin), selon les probabilités définies en (1.1). Puis elles se reproduisent indépendamment, avec la même loi de reproduction que précédemment, et meurent.
— En itérant ce procédé nous obtenons un système de marches aléatoires en milieu aléatoire avec branchement. Remarquons que le processus de branchement sous-jacent, noté Γ, est un processus de Galton–Watson. Nous supposons qu'il est *supercritique*, c'est à dire que le nombre moyen m d'enfants de chaque particule appartient à $(1, +\infty)$. Nous supposons de plus que le nombre d'enfants d'une particule a une variance finie.

Nous nous intéressons plus particulièrement à la position m^*_n de la particule la plus à droite à l'instant n. Le comportement de m^*_n résulte de la compétition de deux phénomènes :

le mouvement d'une particule dans un environnement aléatoire, qui pousse chaque particule vers $-\infty$, et le branchement, qui augmente la probabilité de trouver des particules au voisinage de $+\infty$.

En utilisant les résultats de grandes déviations pour les RWRE, ainsi que des résultats liés aux processus de branchement, nous prouvons le résultat suivant :

Théorème 3.3 *(voir Theorem 1.1, page 49) Supposons que* $\int \log \rho_0(\omega)\eta(d\omega) \geq 0$. *Notons* Γ *le processus de Galton–Watson induit par la marche aléatoire avec branchement, et* $m_c := \exp(I_\eta^q(0))$.
(i) Si $1 < m < m_c$, *alors*

$$\mathbb{P}\left(\limsup_{n\to\infty} \frac{m_n^*}{n} < 0 \mid \Gamma \text{ ne s'éteint pas}\right) = 1.$$

(ii) Si $m > m_c$, *alors*

$$\mathbb{P}\left(\liminf_{n\to\infty} \frac{m_n^*}{n} > 0 \mid \Gamma \text{ ne s'éteint pas}\right) = 1.$$

Notre résultat le plus important est que la particule la plus à droite s'éloigne de l'origine avec une vitesse limite non nulle pour $m \neq m_c$. Noue retrouvons par ailleurs pour notre modèle, par une méthode différente, le résultat de Comets, Menshikov et Popov [16] qui établit sous des hypothèses différentes qu'il existe à partir d'un certain temps des particules dans la partie positive si et seulement si le nombre moyen d'enfants d'une particule est supérieur strictement à m_c.

3.3 Chapitre 4 : Le maximum du temps local d'une diffusion dans un potentiel brownien avec drift

Ce chapitre concerne la diffusion $(X(t),\ t \geq 0)$ dans un potentiel brownien avec drift $W_\kappa(x) := W(x) - \frac{\kappa}{2}x$. On s'intéresse au temps local $(L_X(t,x),\ t \geq 0,\ x \in \mathbb{R})$ de X, et plus particulièrement au maximum du temps local, c'est à dire à $L_X^*(t) := \sup_{x\in\mathbb{R}} L_X(t,x)$. L'étude porte principalement sur les fluctuations de L_X^*. Nous mettons en évidence des comportements surprenants qui contrastent avec ceux observés pour le maximum du temps local d'une RWRE transiente. L'un des principaux résultats est le suivant :

Théorème 3.4 *(voir Theorem 1.7, page 66) Si* $0 < \kappa < 1$,

$$\limsup_{t\to+\infty} \frac{L_X^*(t)}{t} = +\infty \qquad \mathbb{P}\text{–}p.s.$$

Il existe donc $\mathbb{P}$–presque sûrement, dans le cas $0 < \kappa < 1$, une suite $(t_n, x_n) \in (\mathbb{R}_+, \mathbb{R})^{\mathbb{N}}$ telle que

$$\frac{L_X(t_n, x_n)}{t_n} \underset{n\to+\infty}{\longrightarrow} +\infty.$$

Ce résultat est d'autant plus surprenant que la diffusion X est transiente. Il met en évidence la très forte concentration de la diffusion autour des points les plus visités. Ce phénomène contraste avec les résultats correspondants dans le cas des RWRE, obtenus par Gantert et Shi [31]. En effet, le maximum du temps local d'une RWRE à l'instant t est dans ce cas nécessairement majoré par $t/2$.

Par ailleurs, nous déterminons entièrement les fluctuations du maximum du temps local $L_X^*(t)$ dans le cas d'une diffusion transiente à vitesse non nulle, c'est à dire lorsque $\kappa > 1$ (voir Theorem 1.4 page 65 et Theorem 1.6 page 66). Les classes de Lévy au sens « lim sup » pour le maximum du temps local $L_X^*(t)$ sont cette fois-ci en accord avec les résultats obtenus par Gantert et Shi [31] pour les RWRE transientes à vitesse non nulle. Les fluctuations au sens « lim inf » de ce processus sont également établies sous la forme d'une loi du logarithme itéré.

On définit le premier temps d'atteinte du niveau $r \geq 0$ par la diffusion X par

$$H(r) := \inf\{t > 0,\ X(t) = r\}.$$

Nous donnons de plus une caractérisation complète des fluctuations du maximum du temps local au temps $H(r)$.

Théorème 3.5 *(voir Theorem 1.2, page 65) Si $\kappa > 0$, alors*

$$\liminf_{r\to+\infty} \frac{L_X^*(H(r))}{(r/\log\log r)^{1/\kappa}} = 4\left(\frac{\kappa^2}{2}\right)^{1/\kappa} \qquad \mathbb{P}\text{–}p.s.$$

Les classes de Lévy au sens « lim sup » de ce même processus sont données par le test intégral suivant.

Théorème 3.6 *(voir Theorem 1.3, page 65) Soit $a : \mathbb{N} \to \mathbb{R}_+^*$ une fonction croissante. Si $\kappa > 0$, alors*

$$\sum_{n=1}^{\infty} \frac{1}{na(n)} \begin{cases} < +\infty \\ = +\infty \end{cases} \Longleftrightarrow \limsup_{r\to\infty} \frac{L_X^*(H(r))}{[ra(r)]^{1/\kappa}} = \begin{cases} 0 \\ +\infty \end{cases} \qquad \mathbb{P}\text{–}p.s.$$

Par ailleurs, on détermine partiellement le comportement asymptotique au sens « lim inf » du maximum du temps local $L_X^*(t)$ dans les cas $0 < \kappa < 1$ et $\kappa = 1$ (voir Theorem 1.4 page 65 et Theorem 1.5 page 66).

Enfin, on obtient également les classes de Lévy du processus des temps d'atteinte H, qui s'avèrent utiles dans l'étude de $L_X^*(t)$ (voir Theorem 1.8 page 66 et Theorem 1.9 page 67).

Une des conséquences est l'absence de convergence presque-sûre du processus $H(r)/(r \log r)$ dans le cas $\kappa = 1$. Il est à noter par ailleurs que les fluctuations de H sont dans le cas $0 < \kappa < 1$ similaires à celles d'un subordinateur stable d'indice κ.

La preuve de ces résultats se base sur la méthode suivante. Nous commençons par donner, pour $r > 0$ assez grand, une approximation simultanée de $H(r)$ et de $L_X^*(H(r))$ par des fonctionnelles d'un même mouvement brownien, valable avec une grande probabilité annealed. Cette approximation utilise la méthode basée sur la transformation de Lamperti et les processus de Jacobi, présentée par Hu, Shi et Yor [44], qui permet d'étudier les lois asymptotiques de $L_X^*(H(r))$, $H(r)$ et $L_X^*(H(r))/H(r)$.

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Chapitre 2

Some properties of the rate function of quenched large deviations for random walk in random environment

To appear in Markov Processes and Related Fields

In this paper, we are interested in some questions of Greven and den Hollander [3] about the rate function I_η^q of quenched large deviations for random walk in random environment. By studying the hitting times of RWRE, we prove that in the recurrent case, $\lim_{\theta\to 0^+}(I_\eta^q)''(\theta) = +\infty$, which gives an affirmative answer to a conjecture of Greven and den Hollander [3]. We also establish a comparison result between the rate function of quenched large deviations for a diffusion in a drifted Brownian potential, and the rate function for a drifted Brownian motion with the same speed.

Key Words : *Random walk in random environment, Large deviations.*

AMS (2000) **Classification :** 60K37, 60F10, 60J60.

1 Introduction

1.1 Presentation of the model

We consider a collection $(\omega_i)_{i\in\mathbb{Z}}$ of independent and identically distributed random variables taking values in $(0,1)$. A realization of these variables is called an environment. Given an environment $\omega := (\omega_i)_{i\in\mathbb{Z}}$, we consider the random walk $(S_n)_{n\in\mathbb{N}}$ defined by $S_0 = 0$ and

$$P_\omega(S_{n+1} = k | S_n = i) = \begin{cases} \omega_i & \text{if } k = i+1, \\ 1-\omega_i & \text{if } k = i-1, \\ 0 & \text{otherwise.} \end{cases}$$

The process $(S_n)_{n\in\mathbb{N}}$ is called a random walk in random environment, abbreviated RWRE. This model has many applications in physics, see for example Hughes [5]. Let η denote the law of $(\omega_i)_{i\in\mathbb{Z}}$. We call P_ω the quenched law, whereas $\mathbb{P}(.) := \int P_\omega(.)\eta(d\omega)$ is the annealed law. For technical reasons, we assume that there exists an $\varepsilon_0 > 0$ such that

$$\eta(\varepsilon_0 \leq \omega_0 \leq 1-\varepsilon_0) = 1. \tag{1.1}$$

For $i \in \mathbb{Z}$, let $\rho_i = \frac{1-\omega_i}{\omega_i}$. Solomon [11] proved that the RWRE $(S_n)_{n\in\mathbb{N}}$ is η-a.s. recurrent if and only if

$$\int (\log \rho_0)\eta(d\omega) = 0. \tag{1.2}$$

In order to avoid the degenerate case of simple random walk, we assume in the following that

$$\mathrm{Var}(\log \rho_0) := \sigma^2 > 0. \tag{1.3}$$

Sinai [10] showed that in the recurrent case, the random environment considerably slows down the walk. More precisely, he proved that if (1.2) and (1.3) are satisfied, there exists a nondegenerate non–Gaussian random variable b_∞ such that

$$\sigma^2 \frac{S_n}{(\log n)^2} \xrightarrow[n\to+\infty]{\mathcal{L}} b_\infty, \tag{1.4}$$

where $\xrightarrow{\mathcal{L}}$ denotes convergence in law under $\mathbb{P}$.

It is moreover known (Solomon [11]) that the RWRE $(S_n)_{n\in\mathbb{N}}$ satisfies a law of large numbers : there exists $v \in]-1,1[$ such that $\lim_{n\to\infty} S_n/n = v$ $\mathbb{P}$–a.s. In addition, v is deterministic, and is strictly positive if and only if $\int \rho_0\eta(d\omega) < 1$.

The RWRE $(S_n)_{n\in\mathbb{N}}$ satisfies furthermore a quenched large deviation principle with deterministic convex rate function I_η^q (see Greven and den Hollander [3]). This means there exists a nonnegative convex function I_η^q such that η–a.s. for any measurable set A,

$$\begin{aligned}
\liminf_{n\to\infty} \frac{1}{n} \log P_\omega\left(\frac{S_n}{n} \in A\right) &\geq -\inf_{x\in A^\circ} I_\eta^q(x),\\
\limsup_{n\to\infty} \frac{1}{n} \log P_\omega\left(\frac{S_n}{n} \in A\right) &\leq -\inf_{x\in \overline{A}} I_\eta^q(x),
\end{aligned}$$

where A° denotes the interior of A and $\overline{A}$ is the closure of A. For more details on the shape of the rate function, see Comets et al. [2], who also proved a large deviation principle for $(s_n)_{n\in\mathbb{N}}$ under the annealed law $\mathbb{P}$.

For more details on RWRE, we refer to Zeitouni [13].

1.2 Results

In this paper, we are interested in some questions raised by Greven and den Hollander about quenched large deviations for RWRE. First, we answer their Open problem 2 (see [3],

p. 1389 ; see also den Hollander [4] p. 80). A central limit theorem is generally related to the behaviour of $I''(0)$, where I is the rate function considered. Since $(S_n)_{n\in\mathbb{N}}$ has a subdiffusive behaviour (see (1.4)), Greven and den Hollander conjecture that I_η^q has an infinite curvature at 0. We prove that

Theorem 1.1 *Under* (1.1), (1.2) *and* (1.3), *the rate function I_η^q for quenched large deviations of the RWRE satisfies*

$$\lim_{\theta\to 0^+}(I_\eta^q)''(\theta) = +\infty. \tag{1.5}$$

We mention that the corresponding problem for Brox–type diffusions (see Brox [1]), for which the rate functions can be explicitly computed, has already been solved by Taleb (see [12]).

In order to prove Theorem 1.1, it is useful to study the hitting times of $(S_n)_{n\in\mathbb{N}}$. Let us define, for $a \in \mathbb{Z}$,

$$\tau_a := \inf\{n > 0,\ S_n = a\}.$$

We show the following estimate :

Proposition 1.2 *For each $\alpha \in \mathbb{R}_+^*$,*

$$\mathbb{E}(\tau_1^\alpha e^{-r\tau_1}) = \left(\frac{1}{r}\right)^{\alpha+o(1)}, \qquad r \to 0^+.$$

We are also interested in Open problem 3 of Greven and den Hollander ([3], p. 1389) : they conjectured that in the case $\int \rho_0\eta(d\omega) < 1$ (i.e., $v > 0$), the quenched rate function I_η^q of the RWRE is smaller than the rate function of the simple random walk on $\mathbb{Z}$ with the same speed v. That is, they conjectured that $\forall x > v,\ I_\eta^q(x) < \widehat{I}_{\langle\rho\rangle}(x)$, where $\widehat{I}_v$ is the rate function of a usual nearest neighbour random walk with speed v.

Unfortunately, we have not been able to answer this question, but we solve the corresponding problem for Brox–type diffusion (see Brox, [1]). For $\kappa \geq 0$, we define the random potential

$$W_\kappa(x) := W(x) - \frac{\kappa}{2}x,$$

where $(W(x), x \in \mathbb{R})$ is a standard two-sided Brownian motion. We consider a diffusion X in the random potential W_κ, which is defined as the solution to the formal stochastic differential equation

$$\begin{cases} \mathrm{d}X(t) = \mathrm{d}\beta(t) - \frac{1}{2}W'(X(t))\mathrm{d}t, \\ X(0) = 0, \end{cases}$$

where $(\beta(t), t \geq 0)$ is a Brownian motion independent of W. More precisely, X is a diffusion process whose conditional generator given W_κ is

$$\frac{1}{2}e^{W_\kappa(x)}\frac{\partial}{\partial x}\left(e^{-W_\kappa(x)}\frac{\partial}{\partial x}\right).$$

This diffusion can be considered as the continuous time analogue of RWRE and share many properties with it. See for example Shi [9] for the relations between these two processes. For instance, Kawazu and Tanaka [6] established a law of large numbers for X. That is, $\lim_{t\to\infty} X_t/t = v_\kappa$, where $v_\kappa = \frac{(\kappa-1)^+}{4}$ is > 0 if and only if $\kappa > 1$. Moreover, Taleb [12] proved that X satisfies quenched and annealed large deviation principles. Let J_κ denote the rate function of quenched large deviations of X, which is deterministic. See (5.1) below for more details on J_κ. We compare J_κ with the function

$$J^B_{v_\kappa}(x) := \frac{1}{2}(x - v_\kappa)^2,$$

which is the rate function of large deviations of the drifted Brownian motion $(B_t + v_\kappa t,\ t \in \mathbb{R}_+)$. We prove

Theorem 1.3 *If $\kappa > 1$, then*

$$\forall x > v_\kappa, \qquad J_\kappa(x) < J^B_{v_\kappa}(x).$$

This means that for $\kappa > 1$, large deviations for speeds greater than v_κ are less costly for a diffusion in the random environment W_κ than in the corresponding averaged environment.

Interestingly, we obtain as a by-product an inequality for the modified Bessel functions which might be new :

Proposition 1.4 *Let K_ν be the modified Bessel function of index ν. We have,*

$$\forall \nu > 0,\ \forall y > 0, \qquad \frac{K_\nu(y)}{K_{\nu+1}(y)} < \frac{1}{y}\left(\sqrt{y^2+\nu^2} - \nu\right).$$

The rest of the paper is organized as follows : in Section 2 we build environments E_n for which the hitting time of -1 by $(S_n)_{n\in\mathbb{N}}$, denoted by τ_{-1}, will be large. We give an estimation of τ_{-1} for $\omega \in E_n$ in Section 3. In Section 4, we prove Theorem 1.1 and Proposition 1.2. Finally, Section 5 is devoted to the proofs of Theorem 1.3 and Proposition 1.4.

2 Construction of the event E_n

In this section we build a set of environments E_n, such that $\eta(E_n)$ is not “too small” and that for $\omega \in E_n$, τ_{-1} is almost n (we prove this last assertion in Section 3).

Throughout Sections 2 and 3, we fix an $\varepsilon > 0$. The constants C_i, $0 \le i \le 10$, depend only on η and ε, whereas θ and δ depend only on η. The events E_n, $E_{i,n}$ and $E'_{i,n}$ depend on η and ε, but we omit to write ε.

We give some notation in Subsection 2.1. Subsection 2.2 is devoted to the construction of E_n. We give an estimation of $\eta(E_n)$ in Subsection 2.3, and study some of the properties of $\omega \in E_n$ in Subsection 2.4.

2.1 Some notation

We define the potential V as follows :

Definition 2.1 *Let*

$$V(n) := \sum_{i=1}^{n} \log \rho_i = \sum_{i=1}^{n} \log \frac{1-\omega_i}{\omega_i}, \qquad n \in \mathbb{Z},$$

where by convention, $\sum_{i=1}^{0} x_i = 0$ *and* $\sum_{i=1}^{n} x_i = -x_0 - x_{-1} - \cdots - x_{n+1}$ *if* n *is (strictly) negative.*

We define a valley for the potential (see Sinai, [10]) :

Definition 2.2 *Let* $a < m < b$. (a, m, b) *is a valley if*

$$\begin{array}{ll} \forall a \leq i \leq m, & V(m) \leq V(i) \leq V(a), \\ \forall m \leq i \leq b, & V(m) \leq V(i) \leq V(b). \end{array}$$

Its depth is defined as $\min\{V(a) - V(m), V(b) - V(m)\}$.

2.2 Building E_n

In this subsection, we build a valley $(0, m_n, b_n)$ for the potential V, so that the RWRE will stay for a "good" amount of time in this valley with "large probability".

As $\int (\log \rho_0) \eta(d\omega) = 0$ and $\sigma > 0$, there exists a real number $\delta > 0$ such that

$$\eta(-2\delta \leq \log \rho_0 \leq -\delta) := \exp(-\theta) > 0.$$

Now we set

$$\begin{array}{rclrcl} \varepsilon' & := & \varepsilon\delta, & c_{1,n} & := & \lfloor \varepsilon \log n \rfloor, \\ c_{2,n} & := & \lfloor \log n \rfloor^2, & c_{3,n} & := & \delta c_{1,n}, \\ c_{4,n} & := & (1 - 10\varepsilon') \log n, & \beta & := & \frac{1-9\varepsilon'}{1-10\varepsilon'}, \\ c_{5,n} & := & \frac{\varepsilon'}{2} \log n, & c_{6,n} & := & 2 \log n. \end{array}$$

For $i \in \mathbb{Z}$, define $\widetilde{V}(i) = V(i + c_{1,n}) - V(c_{1,n})$ and $\widehat{V}(i) = \widetilde{V}(i + c_{2,n}) - \widetilde{V}(c_{2,n})$. We consider

$$\begin{array}{rcl} E_{1,n} & := & \left\{ \forall 0 \leq i \leq c_{1,n}, \qquad -2\delta i \leq V(i) \leq -\delta i \right\}, \\ E_{2,n} & := & \{ \widetilde{V}(c_{2,n}) \in [-\beta c_{4,n}, -c_{4,n}] \}, \\ E_{3,n} & := & \left\{ \forall 0 < i \leq c_{2,n}, \qquad \left| \widetilde{V}(i) - \dfrac{i}{c_{2,n}} \widetilde{V}(c_{2,n}) \right| \leq c_{5,n} \right\}, \\ E_{4,n} & := & \{ \widehat{V}(c_{2,n}) \in [c_{6,n}, 2c_{6,n}] \}, \\ E_{5,n} & := & \left\{ \forall 0 < i \leq c_{2,n}, \qquad \left| \widehat{V}(i) - \dfrac{i}{c_{2,n}} \widehat{V}(c_{2,n}) \right| \leq c_{5,n} \right\}. \end{array}$$

Finally, let

$$E_n := E_{1,n} \cap E_{2,n} \cap E_{3,n} \cap E_{4,n} \cap E_{5,n}.$$

When $\omega \in E_n$, we say the environment is "good". On $E_{1,n}$, the potential $V(i)$ decreases almost linearly for $i \in [0, c_{1,n}]$ (this will enable the walk $(S_n)_{n\in\mathbb{N}}$ to go quickly to $c_{1,n}$ before hitting -1 with large probability). On $E_{2,n} \cap E_{3,n}$, V stays within a tunnel of height $2c_{5,n}$, and sinks to $V(c_{1,n}) - c_{4,n}$. On $E_{4,n} \cap E_{5,n}$, V stays within another tunnel of height $2c_{5,n}$, and moves up to positive values. These comments on E_n are represented in the Figure 2.1 (b_n and m_n are defined in Subsection 2.4).

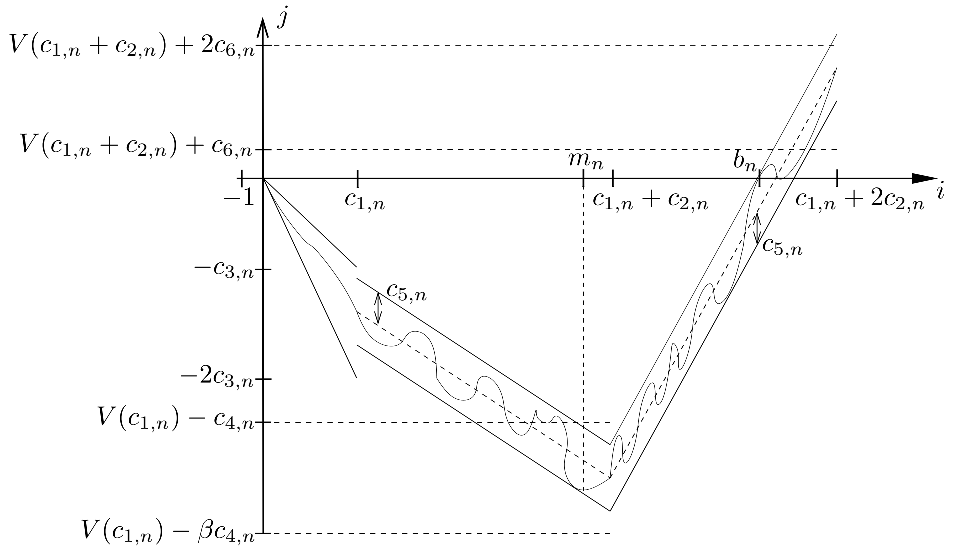

FIG. 2.1 – Example of the potential V for a "good" environment $\omega \in E_n$

2.3 Probability of E_n

Lemma 2.3 *There exists a constant $C_0 > 0$ which depends only on η and ε, such that for n large enough,*

$$\eta(E_n) \geq C_0 n^{-\theta\varepsilon}.$$

Proof : First, observe that

$$\begin{aligned} \eta(E_{1,n}) &\geq \eta\left(\forall 1 \leq i \leq c_{1,n}, \quad -2\delta \leq \log \rho_i \leq -\delta\right) \\ &\geq \exp(-\theta\varepsilon \log n) \geq n^{-\theta\varepsilon}. \end{aligned} \tag{2.1}$$

According to the Komlós–Major–Tusnády strong approximation theorem (see [7]), possibly in an enlarged probability space, there exists a coupling for ω and a standard Brownian motion W, and (strictly) positive constants C_1, C_2 and C_3 such that for all $N \geq 1$,

$$\eta\left(\sup_{1\leq i\leq N} |\widetilde{V}(i) - \sigma W(i)| \geq C_1 \log N\right) \leq \frac{C_2}{N^{C_3}}.$$

Define

$$E'_{6,n} := \left\{ \sup_{1\leq i\leq 3(\log n)^2} |\widetilde{V}(i) - \sigma W(i)| \geq C_1 \log[3(\log n)^2] \right\}.$$

We have,

$$\eta(E'_{6,n}) \leq \frac{C_2}{(3(\log n)^2)^{C_3}}.$$

We then consider the following events :

$$\begin{aligned}
E'_{2,n} &:= \left\{\sigma W(c_{2,n}) \in \left[-\beta c_{4,n} + \frac{\varepsilon'}{4}\log n, -c_{4,n} - \frac{\varepsilon'}{4}\log n\right]\right\}, \\
E'_{3,n} &:= \left\{\forall 0 \leq t \leq c_{2,n}, \quad \left|\sigma W(t) - \frac{t}{c_{2,n}}\sigma W(c_{2,n})\right| \leq c_{5,n} - \frac{\varepsilon'}{4}\log n\right\}.
\end{aligned}$$

We notice by scaling that there exists $C_4 > 0$ such that $\eta[E'_{2,n} \cap E'_{3,n}] \geq 2C_4$ for n large enough. Since $\log[3(\log n)^2] = o(\log n)$, we have for large n,

$$\begin{aligned}
\eta(E_{2,n} \cap E_{3,n}) &\geq \eta[E'_{2,n} \cap E'_{3,n} \cap (E'_{6,n})^c] \\
&\geq \eta(E'_{2,n} \cap E'_{3,n}) - \frac{C_2}{[3(\log n)^2]^{C_3}} \\
&\geq C_4. \qquad (2.2)
\end{aligned}$$

Similarly, there exists a constant $C_5 > 0$ such that

$$\eta(E_{4,n} \cap E_{5,n}) \geq C_5 \qquad (2.3)$$

for n large enough. Since $E_{1,n}$, $E_{2,n} \cap E_{3,n}$ and $E_{4,n} \cap E_{5,n}$ are independent, we obtain Lemma 2.3 by combining (2.1), (2.2) and (2.3). $\square$

2.4 Properties of a "good" environment

Let $\omega \in E_n$. We define the integers b_n and m_n such that

$$\begin{aligned}
b_n &:= \inf\{k \in \mathbb{N},\ k > 0, \quad V(k) \geq 0\}, \\
m_n &:= \inf\{k > 0, \quad V(k) = \inf_{0\leq \ell \leq b_n} V(\ell)\}.
\end{aligned}$$

Note that $(0, m_n, b_n)$ is a valley (in the sense of Definition 2.2) with depth $-V(m_n)$, and that

$$\begin{aligned}
V(m_n) &\in [-2c_{3,n} - \beta c_{4,n} - c_{5,n}, -c_{3,n} - c_{4,n}], \\
m_n &\in \left[c_{1,n} + c_{2,n} - \frac{c_{5,n}c_{2,n}}{c_{4,n}}, c_{1,n} + c_{2,n} + \frac{c_{5,n}c_{2,n}}{c_{6,n}}\right].
\end{aligned}$$

In particular, we have for ε' small enough and n large enough,

$$\begin{array}{rcccl} -\delta + (1-9\varepsilon')\log n & \leq & -V(m_n) & \leq & (1-6\varepsilon')\log n, \\ (1-\varepsilon')(\log n)^2 & \leq & m_n & \leq & (1+\varepsilon')(\log n)^2. \end{array} \tag{2.4}$$

3 Probability that τ_{-1} has a "good" length

This section is devoted to the proof of the following result :

Lemma 3.1 *There exists a constant $C_6 > 0$, depending only on η and ε, such that for all large n,*

$$\forall \omega \in E_n, \qquad P_\omega\left(n^{1-10\varepsilon'} \leq \tau_{-1} < n\right) \geq C_6.$$

In Subsection 3.1, we show that when $\omega \in E_n$, with a large quenched probability, the RWRE goes quickly to the bottom m_n of the valley $(0, m_n, b_n)$ without hitting -1. In Subsection 3.2, we prove that with a large quenched probability, after hitting m_n, the RWRE stays in $\mathbb{N}$ during almost n units of time and then hits -1 for the first time.

3.1 Going to the bottom m_n of the valley

Lemma 3.2 *There exists a constant $C_7 > 0$, depending only on η and ε, such that*

$$\forall \omega \in E_n, \qquad P_\omega(\tau_{m_n} < \tau_{-1}) \geq C_7.$$

Proof : Let $\omega \in E_n$. Since $E_{1,n} \subset E_n$,

$$\sum_{i=0}^{c_{1,n}-1} \exp(V(i)) \leq \sum_{i=0}^{+\infty} \exp(-\delta i) \leq \frac{1}{1-e^{-\delta}}. \tag{3.1}$$

Furthermore,

$$\forall c_{1,n} \leq i \leq m_n, \qquad V(i) \leq -\delta c_{1,n} + c_{5,n} \leq -\frac{\varepsilon'}{2}\log n + \delta.$$

Then, for all large n,

$$0 \leq \sum_{i=c_{1,n}+1}^{m_n-1} \exp(V(i)) \leq 2c_{2,n} n^{-\varepsilon'/2} e^{\delta} \leq 1. \tag{3.2}$$

Accordingly (see Zeitouni [13] p. 196),

$$P_\omega(\tau_{m_n} < \tau_{-1}) = \frac{\exp(V(-1))}{\sum_{k=-1}^{m_n-1}\exp(V(k))} \geq \frac{\frac{\varepsilon_0}{1-\varepsilon_0}}{\frac{1-\varepsilon_0}{\varepsilon_0} + \frac{1}{1-e^{-\delta}} + 1} := C_7 > 0.$$

$\square$

We denote by P_ω^x and E_ω^x the probability and expectation of $(S_n)_{n\in\mathbb{N}}$, starting at site x and conditioned on the environment ω. We have (see Zeitouni, [13], p. 250)

Fact 3.3 *If $a < x < b$,*

$$E_\omega^x(\tau_a \wedge \tau_b) \le \sum_{k=x}^{b-1} \sum_{\ell=a}^{k} \frac{\exp[V(k)-V(\ell)]}{\omega_\ell}. \qquad (3.3)$$

We can now give an upper bound for the hitting time of m_n if the RWRE hits m_n before -1 :

Lemma 3.4 *There exists a constant $C_8 > 0$ such that*

$$\forall \omega \in E_n, \qquad P_\omega(\tau_{m_n} < \tau_{-1} \text{ and } \tau_{m_n} \le n^{4\varepsilon'}) \ge C_8.$$

Proof : According to Fact 3.3 and (2.4), we obtain for $\omega \in E_n$,

$$\begin{aligned} E_\omega(\tau_{-1} \wedge \tau_{m_n}) &\le \sum_{k=0}^{m_n-1} \sum_{\ell=-1}^{k} \frac{\exp[V(k)-V(\ell)]}{\omega_\ell} \\ &\le \frac{1}{\varepsilon_0}(m_n+1)^2 \exp(c_{3,n} + 2c_{5,n}) \\ &\le \frac{1}{\varepsilon_0}[(1+\varepsilon')(\log n)^2 + 1]^2 n^{2\varepsilon'} \le n^{3\varepsilon'}. \end{aligned}$$

Now, by Chebyshev's inequality,

$$P_\omega(\tau_{-1} \wedge \tau_{m_n} \ge n^{4\varepsilon'}) \le n^{-4\varepsilon'} E_\omega(\tau_{-1} \wedge \tau_{m_n}) \le n^{-\varepsilon'}.$$

As a consequence, recalling Lemma 3.2,

$$\begin{aligned} P_\omega(\tau_{m_n} < \tau_{-1} \text{ and } \tau_{m_n} \le n^{4\varepsilon'}) &= P_\omega(\tau_{m_n} < \tau_{-1}) - P_\omega(n^{4\varepsilon'} < \tau_{m_n} < \tau_{-1}) \\ &\ge C_7 - n^{-\varepsilon'} \ge C_7/2 := C_8 \end{aligned}$$

for n large enough. □

3.2 Leaving the valley

First, we give a majoration of the exit time from the valley $(0, m_n, b_n)$.

Lemma 3.5 *There exists a constant $C_9 > 0$, depending only on η and ε, such that*

$$\forall \omega \in E_n, \qquad P_\omega^{m_n}(\tau_{-1} \le n^{1-5\varepsilon'}) \ge C_9.$$

Proof : Let $\omega \in E_n$. The probability to leave the valley $(0, m_n, b_n)$ on the left is

$$\begin{aligned} P_\omega^{m_n}(\tau_{-1} < \tau_{b_n}) &= \frac{1}{\frac{\sum_{k=-1}^{m_n-1}\exp(V(k))}{\sum_{k=m_n}^{b_n-1}\exp(V(k))} + 1} \\ &\geq \frac{1}{\left(1 + \frac{1}{1-e^{-\delta}} + \frac{1-\varepsilon_0}{\varepsilon_0}\right)\frac{1-\varepsilon_0}{\varepsilon_0} + 1} := 2C_9, \end{aligned} \tag{3.4}$$

due to (3.1) and (3.2), and since $\exp(V(b_n)) \geq 1$.

Moreover, Fact 3.3 gives (by symmetry), recalling (2.4),

$$\begin{aligned} E_\omega^{m_n}(\tau_{b_n} \wedge \tau_{-1}) &\leq \sum_{k=0}^{m_n}\sum_{\ell=k}^{b_n} \frac{\exp[V(k-1) - V(\ell-1)]}{\varepsilon_0} \\ &\leq \frac{1}{\varepsilon_0}(b_n+1)^2 \frac{1-\varepsilon_0}{\varepsilon_0}\exp[V(0) - V(m_n)] \\ &\leq (3\log^2 n)^2 \exp[(1-6\varepsilon')\log n]\varepsilon_0^{-2} \\ &\leq n^{1-11\varepsilon'/2} \end{aligned}$$

for n large enough. Then Chebyshev's inequality yields

$$P_\omega^{m_n}(n^{1-5\varepsilon'} < \tau_{b_n} \wedge \tau_{-1}) \leq n^{-\varepsilon'/2}.$$

Consequently, for all environments $\omega \in E_n$, recalling (3.4),

$$\begin{aligned} P_\omega^{m_n}(\tau_{-1} \leq n^{1-5\varepsilon'}) &\geq P_\omega^{m_n}(\tau_{-1} \leq \tau_{b_n} \text{ and } \tau_{-1} \leq n^{1-5\varepsilon'}) \\ &\geq 2C_9 - n^{-\varepsilon'/2} \geq C_9 \end{aligned}$$

for n large enough. □

Now we give a lower bound for τ_{-1}.

Lemma 3.6 *We have,*

$$\inf_{\omega \in E_n} P_\omega^{m_n}(\tau_{-1} > n^{1-10\varepsilon'}) \underset{n\to\infty}{\longrightarrow} 1.$$

Proof : Let $\omega \in E_n$. To establish Lemma 3.6, we use another argument of Sinai's proof. When the RWRE is located at $m_n - 1$, the probability that it hits -1 before going to m_n is

$$\begin{aligned} P_\omega^{m_n-1}(\tau_{-1} < \tau_{m_n}) &= \frac{\exp(V(m_n-1))}{\sum_{k=-1}^{m_n-1}\exp(V(k))} \\ &\leq \exp(V(m_n-1)) \\ &\leq \left(\frac{1-\varepsilon_0}{\varepsilon_0}\right)\frac{e^\delta}{n^{1-9\varepsilon'}}, \end{aligned}$$

due to (2.4). Similarly, we have

$$P_\omega^{m_n+1}(\tau_{b_n} < \tau_{m_n}) \le \left(\frac{1-\varepsilon_0}{\varepsilon_0}\right) \frac{e^\delta}{n^{1-9\varepsilon'}} := \frac{C_{10}}{n^{1-9\varepsilon'}}.$$

As the RWRE is recurrent, we can consider the $\lfloor n^{1-10\varepsilon'} \rfloor$ first excursions away from m_n, which are independent under P_ω. More precisely, let us define recursively

$$\begin{cases} \tau_{m_n}^{(1)} & := \quad \tau_{m_n}, \\ \tau_{m_n}^{(k+1)} & := \quad \inf\{\ell > \tau_{m_n}^{(k)}, \quad S_\ell = m_n\}, \quad k \ge 1, \end{cases}$$

and consider the set

$$E_{7,n} := \left\{\forall 1 \le k \le \lfloor n^{1-10\varepsilon'} \rfloor, \quad \tau_{m_n}^{(k)} < \tau_{-1} \wedge \tau_{b_n}\right\}.$$

We obtain

$$P_\omega^{m_n}(E_{7,n}^c) \le \lfloor n^{1-10\varepsilon'} \rfloor P_\omega^{m_n \pm 1}(\tau_{-1} \wedge \tau_{b_n} < \tau_{m_n}) \le C_{10} n^{-\varepsilon'}.$$

Now, on $E_{7,n}$, the RWRE $(S_i)_{i \ge \tau_{m_n}}$ stays in $[0, b_n]$ during the first $\lfloor n^{1-10\varepsilon'} \rfloor$ excursions away from m_n, hence $\tau_{-1} > n^{1-10\varepsilon'}$. Therefore,

$$\forall \omega \in E_n, \qquad P_\omega^{m_n}(\tau_{-1} > n^{1-10\varepsilon'}) \ge P_\omega^{m_n}(E_{7,n}) \ge 1 - C_{10} n^{-\varepsilon'}.$$

□

Combining Lemmas 3.5 and 3.6, we get

$$\forall \omega \in E_n, \qquad P_\omega^{m_n}(n^{1-10\varepsilon'} < \tau_{-1} \le n^{1-5\varepsilon'}) \ge \frac{C_9}{2}$$

for n large enough. Recalling Lemma 3.4, this ends the proof of Lemma 3.1. □

4 Proofs of Theorem 1.1 and Proposition 1.2

In this section, we use the results of the previous sections to prove Theorem 1.1 and Proposition 1.2.

4.1 Proof of Proposition 1.2

Let $\alpha \in \mathbb{R}_+^*$ and

$$M_\alpha := \sup_{x \in \mathbb{R}_+} (x^\alpha e^{-x}) \in (0, +\infty).$$

Then,

$$\forall r > 0, \qquad \mathbb{E}[\tau_{-1}^\alpha \exp(-r\tau_{-1})] \le M_\alpha r^{-\alpha}. \tag{4.1}$$

Now we give a lower bound for $E_\omega[\tau_{-1}^\alpha \exp(-r\tau_{-1})]$. For any $0 < a < 1$ and any ω,

$$E_\omega[\tau_{-1}^\alpha \exp(-\tau_{-1}/n)] \geq e^{-1} n^{\alpha a} P_\omega(n^a \leq \tau_{-1} \leq n).$$

Thus, by Lemma 3.1, for any $\varepsilon > 0$, taking $a = 1 - 10\varepsilon' = 1 - 10\delta\varepsilon$,

$$\forall \omega \in E_n, \qquad E_\omega[\tau_{-1}^\alpha \exp(-\tau_{-1}/n)] \geq C_6 e^{-1} n^{\alpha(1-10\delta\varepsilon)}.$$

Integrating this inequality on E_n, and in view of Lemma 2.3, we get, for all large n,

$$\mathbb{E}[\tau_{-1}^\alpha \exp(-\tau_{-1}/n)] \geq C_6 C_0 e^{-1} n^{\alpha(1-10\delta\varepsilon)-\theta\varepsilon}.$$

Since $\varepsilon > 0$ can be arbitrary small, this, together with (4.1), yields

$$\mathbb{E}(\tau_{-1}^\alpha e^{-r\tau_{-1}}) = \left(\frac{1}{r}\right)^{\alpha+o(1)}, \qquad r \to 0^+.$$

By symmetry, we can replace τ_1 by τ_{-1}, which gives Proposition 1.2. □

4.2 Proof of Theorem 1.1

It is known (see den Hollander [4], p. 80), that (1.5) is equivalent to

$$\lim_{r\to 0^-} \frac{[\log \lambda]''(r)}{\{[\log \lambda]'(r)\}^3} = 0, \tag{4.2}$$

where

$$\log \lambda(r) = \mathbb{E}[\log E_\omega(e^{r\tau_1})]. \tag{4.3}$$

Note that

$$\begin{aligned} f(r) &:= \frac{[\log \lambda]''(r)}{\{[\log \lambda]'(r)\}^3} \\ &\leq \frac{\mathbb{E}\left(\frac{E_\omega(\tau_1^2 e^{r\tau_1})}{E_\omega(e^{r\tau_1})}\right)}{\left[\mathbb{E}\left(\frac{E_\omega(\tau_1 e^{r\tau_1})}{E_\omega(e^{r\tau_1})}\right)\right]^3} := g(r). \end{aligned}$$

Moreover, due to assumption (1.1), we have, for all $-1 < r < 0$ and for all environments ω,

$$\varepsilon_0 e^{-1} \leq \omega_0 e^r \leq E_\omega(\exp(r\tau_1)) \leq 1.$$

As a consequence, for $-1 < r < 0$,

$$g(r) \leq \frac{e}{\varepsilon_0} \frac{\mathbb{E}[\tau_1^2 \exp(r\tau_1)]}{\{\mathbb{E}[\tau_1 \exp(r\tau_1)]\}^3} := h(r).$$

Furthermore, $f(r) \geq 0$ (by the Cauchy–Schwarz inequality). Now, according to Proposition 1.2,

$$h(r) = |r|^{1+o(1)} \underset{r\to 0^-}{\longrightarrow} 0,$$

which proves (4.2) and thus Theorem 1.1. □

5 Comparison between rate functions

In this section we consider the diffusion X in the random potential W_κ and assume $\kappa > 1$. In this case, $v_\kappa = (\kappa - 1)/4$. We know (see Taleb [78]) that the rate function J_κ of quenched large deviations for X can be written as $J_\kappa(x) = xI_\kappa(1/x)$ for $x > 0$, where

$$I_\kappa(u) = \sup_{\lambda \geq 0}(\Gamma_\kappa(\lambda) - \lambda u), \tag{5.1}$$

and Γ_κ can be expressed in terms of modified Bessel functions (see (5.2) below).

Let

$$\phi_{v_\kappa}(\lambda) := \sqrt{2\lambda + {v_\kappa}^2} - v_\kappa.$$

We first show that $\Gamma_\kappa(\lambda) < \phi_{v_\kappa}(\lambda)$ for large λ. Then we use a differential equation satisfied by Γ_κ to prove that this inequality is true on $\mathbb{R}_+^*$. Finally, we prove Theorem 1.3 and Proposition 1.4.

5.1 Study in the neighbourhood of $+\infty$

According to Taleb (we mention that in Taleb [12], p. 1178, the expression $F_\kappa(\lambda)$ should be $2(2\lambda)^{\kappa/2}K_\kappa[4\sqrt{2\lambda}]$, see for instance Magnus et al., [8] p. 85 ; this misprint has no consequence on the results of [12]), we have

$$\forall \lambda \geq 0, \qquad \Gamma_\kappa(\lambda) = \sqrt{2\lambda}\frac{K_{\kappa-1}(4\sqrt{2\lambda})}{K_\kappa(4\sqrt{2\lambda})}. \tag{5.2}$$

Using the "series of the Hankel type" (see Magnus et al. [8], p. 139), we obtain

$$\Gamma_\kappa(\lambda) = \sqrt{2\lambda} - \frac{1}{4}\left(\kappa - \frac{1}{2}\right) + O\left(\frac{1}{\sqrt{\lambda}}\right) \qquad \lambda \to +\infty. \tag{5.3}$$

This yields

$$\Gamma_\kappa(\lambda) - \phi_{v_\kappa}(\lambda) \underset{\lambda \to +\infty}{\longrightarrow} -\frac{1}{8}.$$

Consequently, there exists $B > 0$, such that

$$\forall \lambda \geq B, \qquad \Gamma_\kappa(\lambda) < \phi_{v_\kappa}(\lambda). \tag{5.4}$$

5.2 Using a differential equation

According to Taleb [12], Γ_κ is a solution of the differential equation $xy' - 2y^2 - \kappa y = -4x$ on $(0, +\infty)$. It is natural to introduce

$$A(x) := x\phi'_{v_\kappa}(x) - 2\phi^2_{v_\kappa}(x) - \kappa\phi_{v_\kappa}(x) + 4x = \frac{-x - v_\kappa^2 + v_\kappa\sqrt{2x + v_\kappa^2}}{\sqrt{2x + v_\kappa^2}}. \tag{5.5}$$

In particular, $A(x) < 0$ for all $x > 0$.

Let us consider the set

$$E := \{x > 0, \quad \Gamma_\kappa(x) \geq \phi_{v_\kappa}(x)\}.$$

We prove that $E = \emptyset$. Indeed, let us assume that $E \neq \emptyset$. According to (5.4), $E \cap [B, +\infty) = \emptyset$. Consequently, E would have a supremum $x_0 \in (0, B]$. By continuity, $\Gamma_\kappa(x_0) = \phi_{v_\kappa}(x_0)$. Now, (5.5) would yield

$$\begin{aligned}
\phi'_{v_\kappa}(x_0) &= \frac{1}{x_0}[A(x_0) + 2\phi^2_{v_\kappa}(x_0) + \kappa\phi_{v_\kappa}(x_0) - 4x_0] \\
&= \frac{1}{x_0}[A(x_0) + 2\Gamma^2_\kappa(x_0) + \kappa\Gamma_\kappa(x_0) - 4x_0] \\
&= \frac{A(x_0)}{x_0} + \Gamma'_\kappa(x_0) < \Gamma'_\kappa(x_0).
\end{aligned}$$

Consequently, there would exist an $\varepsilon > 0$ such that

$$\forall x \in [x_0, x_0 + \varepsilon], \qquad \phi_{v_\kappa}(x) < \Gamma_\kappa(x).$$

Therefore, $[x_0, x_0 + \varepsilon] \subset E$, which contradicts $x_0 = \sup E$. Hence $E = \emptyset$, which means that

$$\forall \lambda > 0, \qquad \Gamma_\kappa(\lambda) < \phi_{v_\kappa}(\lambda). \tag{5.6}$$

5.3 Proofs of Theorem 1.3 and Proposition 1.4

It is easily seen that

$$\forall \lambda \geq 0, \qquad \inf_{0<u<\frac{1}{v_\kappa}} \left\{ \lambda u + \frac{u}{2}\left(\frac{1}{u} - v_\kappa\right)^2 \right\} = \sqrt{2\lambda + {v_k}^2} - v_\kappa = \phi_{v_\kappa}(\lambda).$$

Thus (5.6) yields

$$\forall 0 < u < \frac{1}{v_\kappa},\ \forall \lambda > 0, \qquad \Gamma_\kappa(\lambda) - \lambda u < \frac{u}{2}\left(\frac{1}{u} - v_\kappa\right)^2. \tag{5.7}$$

Notice that (5.7) remains true for $\lambda = 0$ since $\Gamma_\kappa(0) = 0$. Now, fix $u \in (0, 1/v_\kappa)$. Recalling (5.3), it follows that

$$\Gamma_\kappa(\lambda) - \lambda u \underset{\lambda \to +\infty}{\longrightarrow} -\infty.$$

As the function $\lambda \mapsto [\Gamma_\kappa(\lambda) - \lambda u]$ is continuous on $\mathbb{R}_+$, it has a maximum on, say, $\lambda_u \in \mathbb{R}_+$. Hence, by (5.7),

$$\sup_{\lambda \geq 0}(\Gamma_\kappa(\lambda) - \lambda u) = \Gamma_\kappa(\lambda_u) - \lambda_u u < \frac{u}{2}\left(\frac{1}{u} - v_\kappa\right)^2,$$

which can be written as, recalling (5.1) :

$$\forall 0 < u < \frac{1}{v_\kappa}, \qquad I_\kappa(u) < \frac{u}{2}\left(\frac{1}{u} - v_\kappa\right)^2.$$

This is equivalent to

$$\forall x > v_\kappa, \qquad J_\kappa(x) = x I_\kappa\left(\frac{1}{x}\right) < J^B_{v_\kappa}(x) = \frac{1}{2}(x - v_\kappa)^2,$$

proving Theorem 1.3.

We notice that (5.6) can be written in terms of modified Bessel functions, using (5.2), which gives Proposition 1.4.

5.4 Remarks

Recall that the rate function of large deviations of the standard Brownian motion is $x \mapsto x^2/2$. By the same arguments as in the case $\kappa > 1$, we obtain for the transient case with zero speed ($0 < \kappa \leq 1$),

Proposition 5.1 *(zero speed case),*
If $\kappa \in (0, 1/2)$, *then* $\forall x > 0,\ J_\kappa(x) > x^2/2$ *;*
If $\kappa = 1/2$, *then* $\forall x > 0,\ J_\kappa(x) = x^2/2$ *;*
If $\kappa \in (1/2, 1]$, *then* $\forall x > 0,\ J_\kappa(x) < x^2/2$.

(The case $\kappa = 1/2$ was obtained by Taleb, [12]).

We also notice that Proposition 1.4 together with the formula $K_{\nu-1}(z) - K_{\nu+1}(z) = -\frac{2\nu}{z}K_\nu(z)$ also give a lower bound for $K_\nu/K_{\nu+1}$:

$$\forall \nu > 0, \forall y > 0, \qquad \frac{K_\nu(y)}{K_{\nu+1}(y)} > \frac{1}{y}\left[\frac{y^2}{\sqrt{y^2 + (\nu+1)^2} - (\nu+1)} - 2(\nu+1)\right].$$

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Chapter 3

The speed of a branching system of random walks in random environment

We study a branching system of random walks in random environment. Particles reproduce with a fixed reproduction law, and move as one-dimensional random walks in a common random environment.

We assume that the branching mechanism is supercritical with mean $m > 1$, and that the law of the random environment drives a random walk to $-\infty$.

We study the location m_n^* of the rightmost particle. Our main result shows that (conditionally on the survival of the branching process), $\liminf_{n\to+\infty} m_n^*/n > 0$ if m is greater than a certain critical value m_c, and $\limsup_{n\to+\infty} m_n^*/n < 0$ if m is less than the critical value m_c.

Key Words: *Random walk in random environment, branching random walk, Galton–Watson tree.*

AMS (2000) **Classification:** 60J80, 60K37.

1 Introduction

Let $(\omega_i)_{i\in\mathbb{Z}}$ be a collection of independent and identically distributed random variables, taking values in $(0,1)$. For any realization of the environment $\omega := (\omega_i)_{i\in\mathbb{Z}}$, we define a random walk $(X_n)_{n\in\mathbb{N}:=\{0,1,2,\ldots\}}$ which is a Markov chain with $X_0 = 0$ such that for $n \geq 0$ and $i \in \mathbb{Z}$,

$$P_\omega(X_{n+1} = i+1 | X_n = i) = \omega_i, \tag{1.1}$$

$$P_\omega(X_{n+1} = i-1 | X_n = i) = 1-\omega_i. \tag{1.2}$$

We call P_ω the *quenched* law. If η denotes the law of the environment $(\omega_i)_{i\in\mathbb{Z}}$, we also let

$$\mathbb{P}(.) := \int P_\omega(.)\eta(d\omega),$$

and call the resulting law the *annealed* law. This model is known as "random walk in random environment", abbreviated as RWRE. It exhibits some unusual properties, in both annealed and quenched cases (see for example Zeitouni [16]).

For technical reasons, we furthermore assume that there exists $\delta > 0$ such that $\omega_0 \in (\delta, 1-\delta)$ η–a.s.

Let $\rho_i := \frac{1-\omega_i}{\omega_i}$, $i \in \mathbb{Z}$. Solomon [15] proved that the RWRE is η–a.s. recurrent if and only if $\int \log \rho_0(\omega)\eta(d\omega) = 0$. In the transient case, $\lim_{n\to\infty} X_n = +\infty$ η–a.s. if $\int \log \rho_0(\omega)\eta(d\omega) < 0$, and $\lim_{n\to\infty} X_n = -\infty$ η–a.s. if $\int \log \rho_0(\omega)\eta(d\omega) > 0$.

Without loss of generality, we assume that $\int \log \rho_0(\omega)\eta(d\omega) \geq 0$ (otherwise, we only have to replace $(\omega_i)_{i\in\mathbb{Z}}$ by $(1-\omega_i)_{i\in\mathbb{Z}}$).

We are interested in a model of *branching system* of random walks in random environment, where the particles reproduce with a fixed reproduction law, but move as random walks in random environment. This is the analogue in random environment of the model of branching random walks; the latter being well studied in the literature, see for example Révész [14]. We mention that our model is very much *different* from the so-called "branching random walk in random environment" (see for example Greven and den Hollander [8]), where the reproduction law of the particles depends on their locations while the transition probabilities are the same everywhere.

Here is a description of our model. For each given environment, the particle system behaves like this:
— At time $n = 0$ there is only one particle, located at 0.
— At time $n = 1$ the particle moves to 1 with probability ω_0, or to -1 with probability $1-\omega_0$. Arriving at the new location, it gives birth to an amount of k offspring with probability p_k, and dies.
— At time $n = 2$ each particle moves independently (to a neighbouring site), according to the probabilities in (1.1) and (1.2). Then it produces new offspring independently, with the same reproduction law as before, and dies.
— Iterating this procedure we obtain a branching system of random walks in random environment.

We notice that the branching process, denoted by Γ, is a Galton–Watson process. We assume that it is *supercritical*, i.e., the expected amount of offspring of each particle, denoted by m, lies in $(1, +\infty)$. We also assume that the amount of offspring has a finite variance σ^2. We focus on this model throughout the paper. A similar model has already been studied by Comets et al. [5] on the half line $\mathbb{N}$. See also Comets et al. [6] for multidimensional random walks in random environment.

Our model can be expressed in terms of a tree indexed process. See Section 5 for more details. See also Peres [13], chap. 18 for an account on tree indexed processes.

In the case $\int \log \rho_0(\omega)\eta(d\omega) > 0$, there is for our model a competition between the environment, pushing the particles to $-\infty$, and the branching process, which creates new particles and then increases the possibility that some particles go very far on the right. It is therefore natural to study the asymptotic behaviour of the rightmost particle.

Let $\omega_{max} := \max\{x,\ x \in \text{Supp } \omega_0\}$ and

$$m_c := \begin{cases} 1 & \text{if } \frac{1}{2} \in \text{Supp } \omega_0, \\ \frac{1}{2}[\omega_{max}(1-\omega_{max})]^{-1/2} & \text{if Supp } \omega_0 \subset [\delta, \frac{1}{2}). \end{cases} \tag{1.3}$$

Moreover, notice that $m_c = \exp(I_\eta^q(0))$, and that m_c depends only on the law of the environment.

The main result of this paper is the existence of a non-zero speed for the rightmost particle if $m \neq m_c$. More precisely, $\mathbb{P}$–almost surely when the system of particles survives,

(i) if $m < m_c$, then the rightmost particle goes to $-\infty$ with a negative speed;

(ii) if $m > m_c$, then the rightmost particle goes to $+\infty$ with a positive speed.

In the model of Comets et al. [5], the reproduction law also depends on the location, and each particle has almost surely at least one offspring. Hence the branching system always survives. Their result shows that m_c is a critical value for our model in this case. If $m > m_c$, there is infinitely often at least one particle with positive location, and if $m \leq m_c$, there is no particle in $\mathbb{N}$ at time n for n large enough.

By proving Theorem 1.1 below, we show again as a by-product that m_c is a critical value in our setting. Notice that our proof hinges upon a different method, using in particular large deviations for random walks in random environment. In our model, the reproduction law is fixed and does not depend on the location. However, we allow the particles to have no offspring.

Let m_n^* denote the location of the rightmost particle at time n.

Theorem 1.1 *Assume $\int \log \rho_0(\omega)\eta(d\omega) \geq 0$, and let Γ be the Galton–Watson process governing the branching system.*
(i) If $1 < m < m_c$, then

$$\mathbb{P}\left(\limsup_{n\to\infty} \frac{m_n^*}{n} < 0 \mid \Gamma \text{ survives}\right) = 1.$$

(ii) If $m > m_c$, then

$$\mathbb{P}\left(\liminf_{n\to\infty} \frac{m_n^*}{n} > 0 \mid \Gamma \text{ survives}\right) = 1.$$

In some cases, we have $m_c = 1$, which means we are always in situation (ii). It is for example the case when the RWRE is recurrent (i.e., if $\int \log \rho_0(\omega)\eta(d\omega) = 0$). See Section 5 for remarks on the case $m = m_c$.

The article is organized as follows. In Section 2, we give the precise mathematical formulation of the model and present some preliminary results. The proof of the theorem in the case $m < m_c$ is provided in Section 3. Section 4, which is the heart of this paper, consists of the proof for the case $m > m_c$. Finally, Section 5 is devoted to some additional comments.

2 Preliminaries

This section is divided into two parts. The first one consists of a formal definition of the model, whereas the second one deals with large deviations for RWRE.

2.1 Precise formulation of the model

We now give a more formal definition of the model and precise some notation.

Recall that the random variables ω_i are assumed to be independent and identically distributed, with joint distribution η.

Following Révész [14] (who studied the case of usual branching random walks), we introduce the process $(\lambda(x,n))_{x\in\mathbb{Z},n\in\mathbb{N}}$, where for every $x\in\mathbb{Z}$ and $n\in\mathbb{N}$, $\lambda(x,n)$ denotes the number of particles located at x at time n. With this notation, $m_n^* = \max\{x\in\mathbb{Z},\ \lambda(x,n)>0\}$. Let $(Z(x,n,\mu),\ x\in\mathbb{Z},\ n\in\mathbb{N},\ \mu\in\mathbb{N})$ be independent random variables, independent of ω, such that

$$\forall(x,n,\mu,k)\in\mathbb{Z}\times\mathbb{N}^3,\quad \mathbb{P}(Z(x,n,\mu)=k)=p_k.$$

It helps to bear in mind that $Z(x,n,\mu)$ stands for the number of children of the μ-th particle born at location x at time n.

Let $(X(x,n,\mu),\ x\in\mathbb{Z},\ n\in\mathbb{N},\ \mu\in\mathbb{N})$ be independent random variables having the common uniform law on $[0,1]$, which are independent of $(\omega_i)_{i\in\mathbb{Z}}$ and $(Z(x,n,\mu),\ x\in\mathbb{Z},\ n\in\mathbb{N},\ \mu\in\mathbb{N})$. Conditionally on the environment, the movement of the μ-th particle born at location x at time n depends only on $X(x,n,\mu)$.

The process $(\lambda(x,n))_{x\in\mathbb{Z},n\in\mathbb{N}}$, satisfies the following relations: $\lambda(0,0)=1$, $\lambda(x,0)=0$ if $x\neq 0$, and if $n>0$ and $x\in\mathbb{Z}$,

$$\begin{aligned}\lambda(x,n) &= \sum_{\mu=1}^{\lambda(x-1,n-1)} \mathbf{1}_{\{X(x-1,n-1,\mu)\le\omega_{x-1}\}} Z(x-1,n-1,\mu)\\ &+ \sum_{\mu=1}^{\lambda(x+1,n-1)} \mathbf{1}_{\{X(x+1,n-1,\mu)>\omega_{x+1}\}} Z(x+1,n-1,\mu).\end{aligned}$$

The branching process consituted of the particles in the system is denoted by Γ, as in the Introduction.

We denote by P_ω the law of the particle system conditionally on the environment, and as for RWRE we define

$$\mathbb{P}(.) := \int P_\omega(.)\eta(d\omega).$$

Moreover, E_ω and $\mathbb{E}$ denote the expectations with respect to P_ω and $\mathbb{P}$ respectively. We also let $\mathbb{P}^x$ and $\mathbb{E}^x$ (respectively P_ω^x and E_ω^x) denote the probability and expectation of the system (respectively conditionally on the environment) when the root is located at x instead of 0.

We need some more notation. For every $(x, y) \in \mathbb{Z}^2$, $\Delta \subset \mathbb{Z}$ and $(n, p) \in \mathbb{N}^2$, let

$$\begin{aligned} p_\omega(x \rightsquigarrow y, n) &:= P_\omega(X_{p+n} = y | X_p = x), \\ p_\omega(x \rightsquigarrow \Delta, n) &:= P_\omega(X_{p+n} \in \Delta | X_p = x). \end{aligned}$$

Moreover, for any environment ω and any branching system of RWRE, we write

$$\mathcal{F}_\omega(N) := \sigma\left(\lambda(x, n),\ x \in \mathbb{Z},\ 0 \le n \le N\right), \qquad N \in \mathbb{N}.$$

For any $x \in \mathbb{Z}$ and any integers $0 \le n \le N$, we define

$$f_\omega(x, N, n) := m^{N-n} \sum_{y \in \mathbb{Z}} \lambda(y, n) p_\omega(y \rightsquigarrow x, N - n).$$

We can now prove a couple of lemmas, which are simple adaptations from results of Révész [14]:

Lemma 2.1 *We have, for $N \in \mathbb{N}$,*

$$\forall\ 0 \le n \le N,\ \forall x \in \mathbb{Z}, \qquad E_\omega(\lambda(x, N) | \mathcal{F}_\omega(n)) = f_\omega(x, N, n) \qquad \mathbb{P}\text{–a.s.} \tag{2.1}$$

Proof: In the case $N = n$, for each $x \in \mathbb{Z}$,

$$\begin{aligned} E_\omega(\lambda(x, N) | \mathcal{F}_\omega(n)) &= \lambda(x, N) \\ &= \sum_{y \in \mathbb{Z}} \lambda(y, N) p_\omega(y \rightsquigarrow x, 0) \\ &= f_\omega(x, N, N). \end{aligned}$$

Thus (2.1) is true when $N = n$ and in particular when $N = n = 0$.

We deal with the general case by induction. Suppose that for a certain $N \in \mathbb{N}$, formula (2.1) is true for all $0 \le n \le N$ and $x \in \mathbb{Z}$. Therefore, if $n < N + 1$ (the case $n = N + 1$ has already been treated), we have $\mathbb{P}$–a.s.,

$$\begin{aligned} & E_\omega(\lambda(x, N+1) | \mathcal{F}_\omega(n)) \\ = \ & E_\omega\left(\sum_{\mu=1}^{\lambda(x-1,N)} \mathbf{1}_{\{X(x-1,N,\mu) \le \omega_{x-1}\}} Z(x-1, N, \mu) \right. \\ & \left. + \sum_{\mu=1}^{\lambda(x+1,N)} \mathbf{1}_{\{X(x+1,N,\mu) > \omega_{x+1}\}} Z(x+1, N, \mu) | \mathcal{F}_\omega(n) \right) \\ = \ & m\omega_{x-1} E_\omega(\lambda(x-1, N) | \mathcal{F}_\omega(n)) \\ & + m(1 - \omega_{x+1}) E_\omega(\lambda(x+1, N) | \mathcal{F}_\omega(n)). \end{aligned} \tag{2.2}$$

Then, by induction, $\mathbb{P}$–a.s.,

$$\begin{aligned}
(2.2) &= m\omega_{x-1}f_\omega(x-1,N,n)+m(1-\omega_{x+1})f_\omega(x+1,N,n)\\
&= mm^{N-n}\sum_{y\in\mathbb{Z}}\left[\omega_{x-1}p_\omega(y\rightsquigarrow x-1,N-n)\right.\\
&\qquad \left.+(1-\omega_{x+1})p_\omega(y\rightsquigarrow x+1,N-n)\right]\lambda(y,n)\\
&= m^{N+1-n}\sum_{y\in\mathbb{Z}}\lambda(y,n)p_\omega(y\rightsquigarrow x,N+1-n)\\
&= f_\omega(x,N+1,n).
\end{aligned}$$

This yields the desired conclusion. □

As a consequence, we obtain

Lemma 2.2 *We have, for $N\in\mathbb{N}$ and $x\in\mathbb{Z}$,*

$$E_\omega(\lambda(x,N)) = m^N p_\omega(0\rightsquigarrow x,N) \qquad \mathbb{P}\text{–a.s.}$$

Proof: For $N\in\mathbb{N}$ and $x\in\mathbb{Z}$,

$$E_\omega(\lambda(x,N)) = E_\omega(\lambda(x,N)|\mathcal{F}_\omega(0)) \qquad \mathbb{P}\text{–a.s.}$$

Then, using Lemma 2.1, we obtain

$$\begin{aligned}
E_\omega(\lambda(x,N)|\mathcal{F}_\omega(0)) &= f_\omega(x,N,0)\\
&= m^N\sum_{y\in Z}\lambda(y,0)p_\omega(y\rightsquigarrow x,N)\\
&= m^N p_\omega(0\rightsquigarrow x,N) \qquad \mathbb{P}\text{–a.s.},
\end{aligned}$$

since at time 0 there is only one particle, which is at location 0. □

2.2 Large deviations

It is known (Solomon [15]) that the RWRE $(X_n)_{n\in\mathbb{N}}$ satisfies a law of large numbers: there exists $v\in[-1,1]$ such that $\lim_{n\to\infty}X_n/n = v$ $\mathbb{P}$–a.s. Under our assumption $\int\log\rho_0(\omega)\eta(d\omega)\geq 0$, we have $v\in[-1,0]$.

The RWRE $(X_n)_{n\in\mathbb{N}}$ satisfies moreover a quenched large deviation principle with deterministic, convex and continuous rate function I^q_η (see Greven and den Hollander, [9]). This means there exists a nonnegative convex function I_η^q such that η–a.s. for any measurable set A,

$$\liminf_{n\to\infty} \frac{1}{n} \log P_\omega\left(\frac{X_n}{n} \in A\right) \geq -\inf_{x\in A^\circ} I_\eta^q(x), \tag{2.3}$$

$$\limsup_{n\to\infty} \frac{1}{n} \log P_\omega\left(\frac{X_n}{n} \in A\right) \leq -\inf_{x\in \overline{A}} I_\eta^q(x), \tag{2.4}$$

where A° denotes the interior of A and $\overline{A}$ is the closure of A. In case $\int \log \rho_0(\omega)\eta(d\omega) > 0$, Comets et al. [4] (Cases E and F in their Proposition 2) gave the shape of the rate function I_η^q. Let $\omega_{min} := \inf\{x,\ x \in \text{Supp } \omega_0\}$. If $\omega_{min} \leq \frac{1}{2} \leq \omega_{max}$, then $I_\eta^q(0) = 0$. If $\omega_{max} < \frac{1}{2}$, then $v < 0$ and $I_\eta^q(v) = 0$, and I_η^q is strictly increasing on $[v, 1]$.

In the recurrent case $\int \log \rho_0(\omega)\eta(d\omega) = 0$, we simply have $v = 0$ and $I_\eta^q(0) = 0$ ([4], Case A of Proposition 2).

The above was proved in [4] under the additional assumption that η is non degenerate (i.e., it is not concentrated on a single point). However, it holds trivially if η is degenerate, which leads to the case of usual random walk.

3 Proof of Theorem 1.1; case $m < m_c$

In this section, we study the case $1 < m < m_c = \exp(I_\eta^q(0))$. This implies that $I_\eta^q(0) > 0$, thus I_η^q is strictly increasing on $(v, 1]$ with $v < 0$. As a consequence, since I_η^q is a continuous function, there exist $\alpha > 0$ and $\varepsilon > 0$ such that

$$\log m < I_\eta^q(-\alpha) - \varepsilon.$$

According to (2.4), we obtain for the RWRE $(X_n)_{n\in\mathbb{N}}$, $\mathbb{P}$–a.s. for n large enough,

$$P_\omega(X_n \geq -\alpha n) \leq \exp\left[-\left(I_\eta^q(-\alpha) - \varepsilon\right) n\right].$$

As a consequence, recalling Lemma 2.2, we have $\mathbb{P}$–a.s. for all large n,

$$\begin{aligned} P_\omega\{\lambda((-\alpha n, +\infty), n) \geq 1\} &\leq E_\omega\left(\sum_{x=-\alpha n}^{\infty} \lambda(x, n)\right) \\ &= m^n P_\omega(X_n \geq -\alpha n) \\ &\leq \exp\{\,[\log m - I_\eta^q(-\alpha) + \varepsilon]n\,\}. \end{aligned}$$

Since $\log m - I_\eta^q(-a) + \varepsilon < 0$, we obtain

$$\sum_{n\geq 0} P_\omega\{\lambda((-\alpha n, +\infty), n) \geq 1\} < \infty \qquad \mathbb{P}\text{–a.s.}$$

By the Borel–Cantelli lemma, P_ω–a.s. for n large enough, there is no particle in $(-\alpha n, +\infty)$. That is, $\mathbb{P}$–a.s., $m_n^* < -\alpha n$ for n large enough (with $\sup \emptyset = -\infty$ by convention), which leads to first part of Theorem 1.1. □

4 Proof of Theorem 1.1; case $m > m_c$

We now consider the case $m > m_c := \exp(I_\eta^q(0))$, and we prove that the rightmost particle in the system goes to $+\infty$ with a positive speed.

The proof is divided into three steps. (a) We first construct a supercritical Galton–Watson tree T whose vertices of the n-th generation are particles which are at a positive location at time nk_ω (k_ω is an integer defined below). When T survives, which occurs with a positive probability, there is an exponential number of particles in $\mathbb{N}$ at times nk_ω, $n \in \mathbb{N}$. (b) Some of the particles originated from T will go far enough. (c) Although T only has a positive survival probability, there are always particles going very far, as long as the branching process Γ survives.

For the sake of clarity, the three parts are presented in distinct subsections.

4.1 Construction of T

The basic idea in the construction goes back to Hammersley [10], Kingman [11], and Biggins [3]. See Peres ([13] Theorem 18.3) for example.

Recall that $\log m > \log m_c = I_\eta^q(0)$, and fix $\varepsilon > 0$ such that

$$\log m > I_\eta^q(0) + \varepsilon.$$

According to (2.3), we have $\mathbb{P}$–a.s.,

$$\exists n_\omega \in \mathbb{N}^*,\ \forall n \geq n_\omega,\quad P_\omega(X_n \geq 0) \geq \exp[\,-(I_\eta^q(0) + \varepsilon)n\,].$$

Fix such an environment ω, and consider our branching system of particles. By Lemma 2.2, for $k \geq n_\omega$,

$$\begin{aligned} E_\omega\left(\sum_{x\in\mathbb{N}} \lambda(x,k)\right) &= m^k P_\omega(X_k \geq 0) \\ &\geq m^k \exp[-(I_\eta^q(0)+\varepsilon)k] \\ &\geq \exp[(\log m - I_\eta^q(0) - \varepsilon)k]. \end{aligned}$$

Since $\log m > I_\eta^q(0) + \varepsilon$, we can fix an *even* integer $k_\omega > 0$ such that

$$E_\omega\left(\sum_{x\in\mathbb{N}} \lambda(x,k_\omega)\right) := \Lambda_\omega > 2. \tag{4.1}$$

We will be working from now with these fixed constants k_ω and Λ_ω.

We now build recursively a sequence of random variables $(Y_n)_{n\in\mathbb{N}}$ such that at each time nk_ω, $n \in \mathbb{N}$, there are at least Y_n particles located in $\mathbb{N}$:

- At time 0, there is only one particle, which is located at 0. We set $Y_0 = 1$.
- Let Y_1 be the number of particles located in $\mathbb{N}$ at time k_ω, that is, $Y_1 = \lambda(\mathbb{N}, k_\omega)$.
- Suppose that at time nk_ω there are at least Y_n particles in $\mathbb{N}$. Let x_1, x_2,..., x_{Y_n} be the locations of Y_n such particles. We only consider these Y_n particles and ignore all the other particles which are possibly surviving at time nk_ω. By a natural coupling argument, it is easily seen that the number of particles located in $\mathbb{N}$ at time $(n+1)k_\omega$ and generated by these Y_n particles, is greater than or equal to the number of particles located in $\mathbb{N}$ at time $(n+1)k_\omega$ and generated by Y_n particles all of which are located at 0 (instead of (x_i)) at time nk_ω. Thus, at time $(n+1)k_\omega$, there are at least Y_{n+1} particles in $\mathbb{N}$, where
$$Y_{n+1} := \sum_{i=1}^{Y_n} X_{n,i}$$
and the variables $(X_{n,i})$ have the same law as Y_1, and are independent (given ω). Moreover, the variables $(X_{n,i})$ are independent of $\mathcal{F}_\omega(nk_\omega)$.

We denote by T the resulting Galton–Watson tree, of which the cardinality of the n-th generation is Y_n. According to (4.1), $E_\omega(Y_1) = \Lambda_\omega > 2$. Thus T is supercritical; its extinction probability is less than 1.

We are now ready to prove the main technical estimate in this subsection.

Lemma 4.1 *We have, $\mathbb{P}$–almost surely,*
$$\lim_{n\to\infty} P_\omega\left(\bigcap_{\ell\geq n}\{Y_\ell \geq 2^\ell\}\right) > 0.$$

Proof: Consider the original particle system. The amount of offspring of each particle has a finite variance. Therefore the total number of particles at time k_ω, denoted by $B(k_\omega)$, satisfies $E_\omega(B(k_\omega)^2) < \infty$. As a consequence, we have $E_\omega((Y_1)^2) < \infty$. Since $E_\omega(Y_1) = \Lambda_\omega > 2$ (see (4.1)), which is greater than 1, there exists a random variable W_ω, satisfying $P_\omega(W_\omega > 0) > 0$ $\mathbb{P}$–a.s., and such that (see Athreya and Ney [1], p. 9),
$$Y_n \underset{n\to\infty}{\sim} (\Lambda_\omega)^n W_\omega \qquad \mathbb{P}\text{–a.s.}$$
Since $\Lambda_\omega > 2$, $(\Lambda_\omega)^n W_\omega \geq 2^n$ for n large enough if $W_\omega > 0$. Accordingly,
$$\lim_{n\to\infty} P_\omega\left(\bigcap_{\ell\geq n}\{Y_\ell \geq 2^\ell\}\right) = P_\omega(W_\omega > 0) > 0 \qquad \mathbb{P}\text{–a.s.}$$
as desired. □

4.2 Particles going to infinity

This subsection is devoted to proving the following lemma.

Lemma 4.2 *Let m_n^* denote as before the location of the rightmost particle of the system at time n. For almost all environment ω, there exists a real number $S_\omega > 0$ such that*

$$P_\omega\left(\liminf_{n\to\infty}\frac{m_n^*}{n} \geq S_\omega\right) > 0. \tag{4.2}$$

Proof: We have, for $n \in \mathbb{N}$, $A \in \mathbb{N}$, and any even integer N, since k_ω is also even,

$$\begin{aligned}
& P_\omega\{\lambda([A,+\infty), nk_\omega + N) = 0 | \mathcal{F}_\omega(nk_\omega)\} \\
= & \prod_{x\in\mathbb{Z}} \prod_{\ell=1}^{\lambda(2x,nk_\omega)} P_\omega^{2x}\{\lambda([A,+\infty), N) = 0\} \\
\leq & \prod_{x\in\mathbb{N}} \left(P_\omega^{2x}\{\lambda([A,+\infty), N) = 0\}\right)^{\lambda(2x,nk_\omega)}.
\end{aligned}$$

By coupling, we have for $x \geq 0$,

$$P_\omega^{2x}\{\lambda([A,+\infty), N) = 0\} \leq P_\omega^{0}\{\lambda([A,+\infty), N) = 0\}.$$

Thus,

$$\begin{aligned}
& P_\omega\{\lambda([A,+\infty), nk_\omega + N) = 0 | \mathcal{F}_\omega(nk_\omega)\} \\
\leq & \prod_{x\in\mathbb{N}} \left(P_\omega^{0}\{\lambda([A,+\infty), N) = 0\}\right)^{\lambda(2x,nk_\omega)} \\
\leq & \left(P_\omega^{0}\{\lambda([A,+\infty), N) = 0\}\right)^{Y_n}.
\end{aligned} \tag{4.3}$$

Let $a \in (0,1)$ and $\varepsilon' > 0$. According to (2.3), there exists $M_\omega \in \mathbb{N}$ such that

$$\forall N \geq M_\omega, \qquad p_\omega(0 \rightsquigarrow [aN,+\infty), N) \geq \exp[-(I_\eta^q(a)+\varepsilon')N] \qquad \mathbb{P}\text{–a.s.}$$

If q_N denotes the probability that the Galton–Watson tree Γ extincts before time N, we notice that for $N \geq M_\omega$,

$$\begin{aligned}
P_\omega^0(\lambda([aN,+\infty), N) = 0) &\leq q_N + (1-q_N)p_\omega(0 \rightsquigarrow (-\infty, aN), N) \\
&\leq q_N + (1-q_N)\{1-\exp[-(I_\eta^q(a)+\varepsilon')N]\} \\
&= 1-(1-q_N)\exp[-(I_\eta^q(a)+\varepsilon')N].
\end{aligned} \tag{4.4}$$

Let $E_1(\omega, n) := \{Y_n \geq 2^n\}$, and notice that $q_N \leq q_\infty \in [0,1)$. As a consequence, on $E_1(\omega, n)$, using (4.3) and (4.4), we obtain for $N \geq M_\omega$,

$$\begin{aligned} & \log P_\omega\{\lambda([aN, +\infty), nk_\omega + N) = 0 | F_\omega(nk_\omega)\} \\ \leq\ & 2^n \log\{1 - (1 - q_N)\exp[-(I_\eta^q(a) + \varepsilon')N]\} \\ \leq\ & -2^n(1 - q_N)\exp[-(I_\eta^q(a) + \varepsilon')N] \\ \leq\ & -(1 - q_\infty)\exp[n\log 2 - (I_\eta^q(a) + \varepsilon')N]. \end{aligned}$$

Let $N_n = 2\lfloor \frac{n \log 2}{4(I_\eta^q(a)+\varepsilon')} \rfloor$. For all large n, we obtain on $E_1(\omega, n)$,

$$P_\omega\{\lambda([aN_n, +\infty), nk_\omega + N_n) = 0 | F_\omega(nk_\omega)\} \leq \exp\left(-(1 - q_\infty)C\exp\left(n\frac{\log 2}{2}\right)\right),$$

where $C > 0$ is a constant. Hence,

$$\begin{aligned} & P_\omega(\{\lambda([aN_n, +\infty), nk_\omega + N_n) = 0\} \cap E_1(\omega, n)) \\ \leq\ & \exp\left(-(1 - q_\infty)C\exp\left(n\frac{\log 2}{2}\right)\right). \end{aligned}$$

Consequently,

$$\sum_{n \in \mathbb{N}} P_\omega(\{\lambda([aN_n, +\infty), nk_\omega + N_n) = 0\} \cap E_1(\omega, n)) < +\infty.$$

By Lemma 4.1, for almost all environment ω there exists $n_\omega \in \mathbb{N}$ such that $P_\omega(E_2(\omega, n_\omega)) > 0$, where $E_2(\omega, n) := \cap_{\ell \geq n} E_1(\omega, \ell)$. By the Borel–Cantelli lemma, we obtain P_ω–a.s. on $E_2(\omega, n_\omega)$, for n large enough,

$$\lambda([aN_n, +\infty), N_n + nk_\omega) \geq 1.$$

Then P_ω–a.s. on $E_2(\omega, n_\omega)$, for all large n, there exists a particle p_n in $[aN_n, +\infty)$ at time $K_n = N_n + nk_\omega$. At any time $\ell \in (K_{n-1}, K_n] \cap \mathbb{Z}$, the ancestor of the particle p_n is located in $[aN_n - (K_n - K_{n-1}), +\infty)$, which is contained in $[S_\omega \ell, +\infty)$ for some constant $S_\omega > 0$ (noticing that $K_n - K_{n-1}$ is bounded). Thus, for all large ℓ, $\lambda([S_\omega \ell, +\infty), \ell) \geq 1$. This means that P_ω–a.s. on $E_2(\omega, n_\omega)$ there are at any large time some particles with average speed greater than S_ω. Since $P_\omega\{E_2(\omega, n_\omega)\} > 0$, this completes the proof of the lemma. $\square$

4.3 End of the proof

For any $S > 0$, we define the event

$$A(S) := \left\{\liminf_{n \to \infty} \frac{m_n^*}{n} \geq S\right\},$$

where $m_n^* = -\infty$ if these is no particle left at time n. Let θ denote the shift operator, given by $(\theta\omega)_i = \omega_{i+1}$. The sequence $\{P_{\theta^{2i}\omega}(A(S))\}_{i\in\mathbb{Z}}$ is a stationary sequence. Moreover, by a simple coupling argument, it is also a nondecreasing sequence. Thus it is constant, i.e. $\mathbb{P}$–a.s.,

$$\forall i \in \mathbb{Z}, \qquad P_{\theta^{2i}\omega}(A(S)) = P_\omega(A(S)).$$

By Lemma 4.2, for almost all environment ω, there exists $S_\omega > 0$ such that $P_\omega(A(S_\omega)) > 0$. We now fix such an ω and set $S := S_\omega$.

Let $1 < r < m$. Let $B(N)$ denote the total number of particles at time N. For $N \in \mathbb{N}$, let

$$E_3(N) := \{B(N) \geq r^N\}.$$

Since Γ is a Galton–Watson tree such that $1 < m < \infty$ and $\sigma^2 < \infty$,

$$\{\Gamma \text{ survives}\} = \liminf_{N\to\infty} E_3(N).$$

We notice that on $E_3(2N)$, with the convention $\sup\emptyset = -\infty$,

$$\begin{aligned}
P_\omega\left(\liminf_{n\to\infty}\frac{m_n^*}{n} < S \mid \mathcal{F}_\omega(2N)\right) &\leq \prod_{x\in\mathbb{Z}}\left(P_\omega^{2x}\left(\liminf_{n\to\infty}\frac{m_n^*}{n} < S\right)\right)^{\lambda(2x,2N)} \\
&= \prod_{x\in\mathbb{Z}}\{1 - P_{\theta^{2x}\omega}[A(S)]\}^{\lambda(2x,2N)} \\
&\leq \{1 - P_\omega[A(S)]\}^{r^{2N}}.
\end{aligned}$$

Consequently, as $P_\omega(A(S)) > 0$,

$$\lim_{N\to\infty} P_\omega\left(\left\{\liminf_{n\to\infty}\frac{m_n^*}{n} < S_\omega\right\} \cap E_3(2N)\right) = 0.$$

Since $\{\Gamma \text{ survives}\} = \liminf_{N\to\infty} E_3(N)$, this yields Part (ii) of Theorem 1.1. $\square$

5 Comments

We present some further remarks in this section.

5.1 Case $m = m_c$

Consider the critical case $m = m_c$. In this case, $I_\eta^q(0) > 0$, $v < 0$, $I_\eta^q(v) = 0$, and I_η^q is strictly increasing on $[v, 1]$. The method used in Section 4 and an indirect proof lead to the following result:

Proposition 5.1 *If $m = m_c$, then*

$$\mathbb{P}\left(\limsup_{n\to\infty} \frac{m_n^*}{n} \leq 0 |\ \Gamma \text{ survives}\right) = 1, \tag{5.1}$$

and

$$\forall k \in \mathbb{N}, \quad E_\omega(\lambda(\mathbb{N}, 2k)) \leq 1 \qquad \mathbb{P}\text{–a.s.} \tag{5.2}$$

Thus there are very few particles (if any) in $\mathbb{N}$ in the critical case. Moreover, Gantert and Müller [7] prove (in a more general setting) that in this critical case, when there is always at least one offspring particle, the system is "transient", that is, $m_n^* \to -\infty$.

Proof: To prove (5.1), we follow the ideas of Section 3. For every $\alpha > 0$, there exists $\varepsilon > 0$ such that

$$\log m = I_\eta^q(0) < I_\eta^q(\alpha) - \varepsilon.$$

This yields

$$\sum_{n\geq 0} P_\omega(\lambda((\alpha n, +\infty), n) \geq 1) < \infty.$$

As in Section 3, this gives (5.1).

Now, let $k \in \mathbb{N}$ and $A_k := \{E_\omega(\lambda(\mathbb{N}, k)) > 1\}$. Suppose that $P_\omega(A_k) > 0$. For $\omega \in A_k$ we can build the tree T as before, with k instead of k_ω, This is a supercritical Galton–Watson tree, since $E_\omega(\lambda(\mathbb{N}, k))) > 1$. Then, $E_\omega([T]_n) \geq W_\omega a_\omega^n$ for n large enough, with $P_\omega(W_\omega > 0) > 0$ and $a_\omega > 1$. Let $\varepsilon > 0$. For $\varepsilon' > 0$ and n large enough,

$$\begin{aligned} E_\omega(\lambda([\varepsilon N, +\infty[, N + nk)) &\geq E_\omega([T]_n) E_\omega(\lambda([\varepsilon N, +\infty[, N)) \\ &\geq W_\omega a_\omega^n \exp(-I_\eta^q(\varepsilon)N - \varepsilon' N) \\ &= exp(-(I_\eta^q(\varepsilon) + \varepsilon')N + (\log a_\omega)n) \end{aligned}$$

If ε' is small enough, for $n = \lfloor \frac{2(I_\eta^q(e)+\varepsilon')}{\log \alpha_\omega} N \rfloor$, and $\beta = \left(\frac{2(I_\eta^q(\varepsilon)+\varepsilon')}{\log a_\omega} k + 1\right)$, we have

$$E_\omega(\lambda([\varepsilon N, +\infty[, \beta N)) \sim_{N\to+\infty} E_\omega\left(\lambda\left([\varepsilon N, +\infty[, N + nk\right)\right) \longrightarrow_{N\to+\infty} +\infty \tag{5.3}$$

But on the other hand, for N large enough and $\varepsilon_2 > 0$ small enough,

$$\begin{aligned} E_\omega(\lambda([\varepsilon N, +\infty[, \beta N)) &\leq m^{\beta N} \exp(-(I_\eta^q(\varepsilon/\beta) - \varepsilon_2)\beta N) \\ &= \exp((I_\eta^q(0) - I_\eta^q(\varepsilon/\beta) + \varepsilon_2)\beta N) \longrightarrow_{N\to\infty} 0. \end{aligned}$$

This contradicts (5.3). Hence, $\mathbb{P}(A_k) = 0$, for every $k \in \mathbb{N}^*$, which gives (5.2). $\square$

5.2 Tree indexed process

Another point of view for the model studied in this paper is to view the system of particles as a tree indexed process $(Y_v)_{v\in\Gamma}$. We define it by the following relations:

$$Y_{\mathbf{o}} = 0,$$

where $\mathbf{o}$ is the root of Γ. Moreover, for all $v \in \Gamma$, if the children of v are denoted by $v_1, v_2, \dots v_n$, we set for all $x \in \mathbb{Z}$,

$$\begin{aligned} P_\omega\left(Y_{v_1} = Y_{v_2} = \cdots = Y_{v_n} = x+1 | \Gamma, Y_v = x\right) &= \omega_x, \\ P_\omega\left(Y_{v_1} = Y_{v_2} = \cdots = Y_{v_n} = x-1 | \Gamma, Y_v = x\right) &= 1-\omega_x. \end{aligned}$$

Furthermore, the locations of the offspring of two different vertices u and v of the same generation of Γ are independent conditionally on Γ, Y_u and Y_v. In this case, the location of the rightmost particle is $m_n^* = \max\{Y_v,\ v \in [\Gamma]_n\}$, where $[\Gamma]_n$ denote the set of the vertices of Γ of the n–th generation. One can formulate the proof of Theorem 1.1 in terms of $(Y_v)_{v\in\Gamma}$.

5.3 Value of the speed

It would be interesting to know, as in the case of usual random walks, the values of the burst speed, cloud speed and sustainable speed of the tree indexed process $(Y_v)_{v\in\Gamma}$ (see Benjamini and Peres [2], Lyons and Pemantle [12]).

5.4 A similar model

Theorem 1.1 holds also for the following model: at time $t = 0$ there is only one particle, located at 0. At time n, each particle reproduces independently with the same law: it gives birth to k offspring with probability p_k and dies. Then each particle moves to a new location according to the transition probabilities $(\omega_i)_{i\in\mathbb{Z}}$. See for example Biggins [3] in the case of usual random walks.

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Chapter 4

Maximum of the local time of a diffusion in a drifted Brownian potential

We study a one-dimensional diffusion process X in a drifted Brownian potential. We are interested in the maximum of its local time, and study its almost sure asymptotic behaviour, which is proved to be different from the behaviour of the maximum local time of the transient random walk in random environment. We also characterize all the upper and lower classes for X, in the sense of Paul Lévy.

KEY WORDS:*Random environment, diffusion in a random potential, maximum local time, Lévy class.*

AMS (2000) Classification: 60K37, 60J60, 60J55, 60F15.

1 Introduction

1.1 Presentation of the model

We consider a diffusion process in random environment, defined as follows. For $\kappa \in \mathbb{R}$, we introduce the random potential

$$W_\kappa(x) := W(x) - \frac{\kappa}{2}x, \qquad x \in \mathbb{R}, \tag{1.1}$$

where $(W(x),\ x \in \mathbb{R})$ is a standard two-sided Brownian motion. We define a diffusion process $(X(t),\ t \geq 0)$ in the random potential W_κ as solution to the formal stochastic differential equation

$$\begin{cases} \mathrm{d}X(t) = \mathrm{d}\beta(t) - \frac{1}{2}W_\kappa'(X(t))\mathrm{d}t, \\ X(0) = 0, \end{cases}$$

where $(\beta(t),\ t \geq 0)$ is a Brownian motion independent of W. More rigorously, X is a diffusion process such that $X(0) = 0$, and whose conditional generator given W_κ is

$$\frac{1}{2}e^{W_\kappa(x)}\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{-W_\kappa(x)}\frac{\mathrm{d}}{\mathrm{d}x}\right).$$

We denote by P_ω the law of X conditionally on the environment W_κ, and call it the quenched law. We also define the annealed law $\mathbb{P}$ as follows:

$$\mathbb{P}(\cdot) := \int P_\omega(\cdot)\mathbb{P}(W_\kappa \in \mathrm{d}\omega).$$

The diffusion X is generally considered as the continuous time analogue of random walks in random environment (RWRE), which have many applications in physics and biology (see e.g. Le Doussal et al. [17]; for an account of general properties of RWRE, we refer to Révész [19] and Zeitouni [30]). Such diffusions, introduced by Schumacher [22] and Brox [6], have been studied for example by Kawazu and Tanaka [15], Hu et al. [13], Mathieu [18], Carmona [7] and Taleb [27]. For a relation between RWRE and the diffusion X, see [22].

In this paper, we are interested in the transient case, that is, we suppose $\kappa \neq 0$. If X is a diffusion in the random potential W_κ, then $-X$ is a diffusion in the random potential $(W_\kappa(-x),\ x \in \mathbb{R})$ which has the same law as $(W_{-\kappa}(x),\ x \in \mathbb{R})$. Hence we may assume without loss of generality that $\kappa > 0$. In this case, $X(t) \to_{t\to+\infty} +\infty$ $\mathbb{P}$–almost surely.

Our goal is to study the almost sure asymptotics of the supremum of the local time of X. Corresponding problems for RWRE have been studied, for example, in Révész ([19], Chapter 29), Gantert and Shi [10], Shi [24], Hu and Shi [11], Andreoletti [1].

1.2 Results

We denote by $(L_X(t,x),\ t \geq 0,\ x \in \mathbb{R})$ the local time of X, which is the jointly continuous process satisfying, for any positive measurable function f,

$$\int_0^t f(X(s))\mathrm{d}s = \int_{-\infty}^{+\infty} f(x)L_X(t,x)\mathrm{d}x, \qquad t \geq 0. \tag{1.2}$$

We are interested in the maximum local time of X at time t, defined as

$$L_X^*(t) := \sup_{x\in\mathbb{R}} L_X(t,x), \qquad t \geq 0.$$

Let

$$H(r) := \inf\{t \geq 0, \quad X(t) > r\}, \qquad r \geq 0 \tag{1.3}$$

be the first hitting time of r by X. We recall that there are three different regimes for H:

Theorem 1.1 *(Kawazu and Tanaka, [15]) When r tends to infinity,*

$$\begin{aligned}
\frac{H(r)}{r^{1/\kappa}} &\xrightarrow{\mathcal{L}} c_0 S_\kappa^{ca}, \qquad 0<\kappa<1,\\
\frac{H(r)}{r\log r} &\xrightarrow{P.} 4, \qquad \kappa=1,\\
\frac{H(r)}{r} &\xrightarrow{a.s.} \frac{4}{\kappa-1}, \qquad \kappa>1,
\end{aligned}$$

where $c_0 = c_0(\kappa) > 0$ is a finite constant, the symbols " $\xrightarrow{\mathcal{L}}$ ", " $\xrightarrow{P.}$ " and " $\xrightarrow{a.s.}$ " denote respectively convergence in law, in probability and almost sure convergence, with respect to the annealed probability $\mathbb{P}$. Moreover, S_κ^{ca} is a completely asymmetric stable variable of index κ, and is a positive variable for $0<\kappa<1$ (see (5.1) for its characteristic function).

The first set of our results gives a precise description of the almost sure asymptotics of L_X^* along the first hitting times.

Theorem 1.2 *For $\kappa > 0$,*

$$\liminf_{r\to+\infty} \frac{L_X^*(H(r))}{(r/\log\log r)^{1/\kappa}} = 4\left(\frac{\kappa^2}{2}\right)^{1/\kappa} \qquad \mathbb{P}\text{–}a.s.$$

Theorem 1.3 *Let $\kappa > 0$. For any positive nondecreasing function $a(\cdot)$, we have*

$$\sum_{n=1}^{\infty} \frac{1}{na(n)} \begin{cases} <\infty \\ =+\infty \end{cases} \iff \limsup_{r\to\infty} \frac{L_X^*(H(r))}{[ra(r)]^{1/\kappa}} = \begin{cases} 0 \\ +\infty \end{cases} \qquad \mathbb{P}\text{–}a.s.$$

If we consider $L_X^*(t)$ instead of $L_X^*(H(r))$, the situation is considerably more complex, and heavily depends on the value of κ. We start with the lower asymptotics of $L_X^*(t)$:

Theorem 1.4 *We have*

$$\begin{aligned}
\liminf_{t\to\infty} \frac{L_X^*(t)}{t/\log\log t} &\le \kappa^2 c_1(\kappa) \qquad \mathbb{P}\text{–}a.s. \quad \text{if } 0<\kappa<1,\\
\liminf_{t\to\infty} \frac{L_X^*(t)}{t/[(\log t)\log\log t]} &\le \frac{1}{2} \qquad \mathbb{P}\text{–}a.s. \quad \text{if } \kappa=1,\\
\liminf_{t\to\infty} \frac{L_X^*(t)}{(t/\log\log t)^{1/\kappa}} &= 4\left(\frac{(\kappa-1)\kappa^2}{8}\right)^{1/\kappa} \qquad \mathbb{P}\text{–}a.s. \quad \text{if } \kappa>1.
\end{aligned}$$

In the case $0 < \kappa < 1$, $c_1(\kappa)$ is given by the variational formula

$$c_1(\kappa) := \inf\left\{ \frac{1}{2}\int_0^1 |\phi'(v)|^2 \mathrm{d}v, \quad \phi \in \mathcal{C}_0, \quad \int_0^1 (1-v)^{1/\kappa-2}|\phi(v)|^2 \mathrm{d}v \geq 1 \right\},$$

where $\mathcal{C}_0$ is the set of continuous functions $\phi : [0,1] \to \mathbb{R}$ such that $\phi(0) = 0$.

Theorem 1.5 *We have, for any $\varepsilon > 0$,*

$$\liminf_{t\to\infty} \frac{L_X^*(t)}{t/[(\log t)^{1/\kappa}(\log\log t)^{(2/\kappa)+\varepsilon}]} \quad = \quad +\infty \qquad \mathbb{P}\text{–}a.s. \quad \text{if } 0 < \kappa \leq 1.$$

In the case $0 < \kappa \leq 1$, Theorems 1.4 and 1.5 give different bounds, for technical reasons. We now turn to the upper asymptotics of $L^*(t)$.

Theorem 1.6 *Let $a(\cdot)$ be a nondecreasing function. If $\kappa > 1$, then*

$$\sum_{n=1}^{\infty} \frac{1}{na(n)} \begin{cases} < +\infty \\ = +\infty \end{cases} \Longleftrightarrow \limsup_{t\to\infty} \frac{L_X^*(t)}{[ta(t)]^{1/\kappa}} = \begin{cases} 0 \\ +\infty \end{cases} \qquad \mathbb{P}\text{–}a.s.$$

Theorem 1.7 *If $0 < \kappa < 1$, then*

$$\limsup_{t\to+\infty} \frac{L_X^*(t)}{t} = +\infty \qquad \mathbb{P}\text{–}a.s.$$

Theorem 1.6 gives, in the case $\kappa > 1$, an integral test which completely characterizes the upper functions of $L^*(t)$, in the sense of Paul Lévy. This is in agreement with a result of Gantert and Shi [10] for RWRE.

Theorem 1.7 tells us that in the case $\kappa < 1$, the maximum local time of X has completely different behaviour from the maximum local time of RWRE (the latter is trivially bounded by $t/2$ for any positive integer t, for example). Such a peculiar phenomenon has already been observed by Shi [24] in the recurrent case, and is even more surprising here since X is transient.

We have not been able to settle the very delicate critical case $\kappa = 1$.

In the proof of Theorems 1.4, 1.5 and 1.7, we will frequently need to study the almost sure asymptotics of the first hitting times $H(\cdot)$. In view of the last part of Theorem 1.1, we only need to study the case $\kappa \in (0, 1]$.

Theorem 1.8 *Let $a(\cdot)$ be a positive nondecreasing function. If $0 < \kappa < 1$, then*

$$\sum_{n=1}^{\infty} \frac{1}{na(n)} \begin{cases} < +\infty \\ = +\infty \end{cases} \Longleftrightarrow \limsup_{r\to\infty} \frac{H(r)}{[ra(r)]^{1/\kappa}} = \begin{cases} 0 \\ +\infty \end{cases} \qquad \mathbb{P}\text{–}a.s.$$

If $\kappa = 1$, the statement holds under the additional assumption that $\limsup_{r\to+\infty}(\log r)/a(r) < \infty$.

Theorem 1.9 *If $0 < \kappa < 1$, then*

$$\liminf_{r\to+\infty} \frac{H(r)}{r^{1/\kappa}/(\log\log r)^{(1/\kappa)-1}} = c_2(\kappa) \qquad \mathbb{P}\text{–}a.s.,$$

where the value of $c_2(\kappa)$ is given in equation (5.9)*.*

If $\kappa = 1$, then

$$\liminf_{r\to+\infty} \frac{H(r)}{r\log r} = 4 \qquad \mathbb{P}\text{–}a.s.$$

It was asked in Hu et al. [13] whether the convergence in probability $H(r)/(r\log r) \to 4$ in Theorem 1.1 in the case $\kappa = 1$ can be strengthened into an almost sure convergence. Theorem 1.8 gives a negative answer.

We observe that in the case $0 < \kappa < 1$, the process $H(\cdot)$ has the same Lévy classes as κ–stable subordinators (see Bertoin [3] p. 92).

Theorems 1.8 and 1.9 can be stated for the process X itself, by means of a standard argument.

The rest of the paper is organized as follows. In Section 2, we give some preliminaries on Bessel processes. We present in Section 3 some technical estimates which will be needed later on; the proof of one of the technical estimates (Lemma 3.3), is postponed until Section 7. Section 4 is devoted to the study of $L_X^*(H(r))$ and the proofs of Theorems 1.2 and 1.3. In Section 5, we study the Lévy classes for the hitting times $H(r)$ and prove Theorems 1.8 and 1.9. In Section 6, we study $L_X^*[H(r)]/H(r)$ and prove Theorems 1.4–1.7. Finally, Section 7 is devoted to the proof of Lemma 3.3.

Throughout the paper, the letter c with a subscript, denotes unimportant constants that are finite and positive.

2 Preliminaries on Bessel processes

For any Brownian motion $(B(t),\ t \geq 0)$ and $r > 0$, we define the first hitting time

$$\sigma_B(r) := \inf\{t > 0, \quad B(t) = r\}.$$

Moreover, we denote by $(L_B(t,x),\ t \geq 0,\ x \in \mathbb{R})$ the local time of B, i.e., the jointly continuous process satisfying $\int_0^t f(B(s))\mathrm{d}s = \int_{-\infty}^{+\infty} f(x)L_B(t,x)\mathrm{d}x$ for any positive measurable function f. We define the inverse local time of B as

$$\tau_B(a) := \inf\{t \geq 0, \quad L_B(t,0) \geq a\}, \qquad a > 0.$$

Furthermore, for any $\delta \geq 0$ and $x \geq 0$, the unique strong solution of the stochastic differential equation

$$Z(t) = x + 2\int_0^t \sqrt{Z(s)}\mathrm{d}\beta(s) + \delta t,$$

where $(\beta(s), s \geq 0)$ is a Brownian motion, is called a δ–dimensional squared Bessel process starting from x. A δ–dimensional Bessel process starting from x is defined as the (nonnegative) square root of a δ–dimensional squared Bessel process starting from x^2. We recall some important results.

Fact 2.1 *(first Ray–Knight theorem) Consider $r > 0$ and a Brownian motion $(B(t),\ t \geq 0)$. The process $(L_B(\sigma_B(r), r - x),\ x \geq 0)$ is a continuous inhomogeneous Markov process, starting from 0. It is a 2–dimensional squared Bessel process for $x \in [0, r]$ and a 0–dimensional squared Bessel process for $x \geq r$.*

Fact 2.2 *(second Ray–Knight theorem) Fix $r > 0$, and let $(B(t),\ t \geq 0)$ be a Brownian motion. The process $(L_B(\tau_B(r), x),\ x \geq 0)$ is a 0–dimensional squared Bessel process starting from r.*

Fact 2.3 *(Lamperti representation theorem [16]) Consider $W_\kappa(x) = W(x) - \kappa x/2$ as in (1.1), where $(W(x),\ x \geq 0)$ is a Brownian motion. There exists a $(2 - 2\kappa)$–dimensional Bessel process $(\rho(t),\ t \geq 0)$, starting from $\rho(0) = 2$, such that*

$$\exp[W_\kappa(t)/2] = \frac{1}{2}\rho\left(\int_0^t e^{W_\kappa(s)} \mathrm{d}s\right), \qquad t \geq 0.$$

See e.g. Revuz and Yor ([20], chap. XI) for more details about Ray–Knight theorems and Bessel processes.

We also recall the following extension to Bessel processes of Williams' time reversal theorem (see Yor [29], p. 80).

Fact 2.4 *One has, for $\delta < 2$,*

$$(R_\delta(T_0 - s),\ s \leq T_0) \stackrel{\mathcal{L}}{=} (R_{4-\delta}(s),\ s \leq \gamma_a),$$

where $\stackrel{\mathcal{L}}{=}$ denotes equality in law, $(R_\delta(s),\ s \geq 0)$ denotes a δ–dimensional Bessel process starting from $a > 0$, $T_0 := \inf\{s \geq 0,\ R_\delta(s) = 0\}$, $(R_{4-\delta}(s),\ s \geq 0)$ is a $(4 - \delta)$–dimensional Bessel process starting from 0, and $\gamma_a := \sup\{s \geq 0,\ R_{4-\delta}(s) = a\}$.

3 Technical estimates

We start by introducing

$$A(x) := \int_0^x e^{W_\kappa(y)} \mathrm{d}y, \qquad x \in \mathbb{R},$$

which is a scaling function of X. We observe that, since $\kappa > 0$, $A(x) \to A_\infty < \infty$ when $x \to +\infty$.

For technical reasons, we have to introduce the random function F as follows. Fix $r > 0$. Since the function $x \mapsto A_\infty - A(x)$ is almost surely continuous and (strictly) decreasing, there exists a unique $F(r) \in \mathbb{R}$, depending only on the process W_κ, such that

$$A_\infty - A(F(r)) = \exp(-\kappa r/2) = \delta(r). \tag{3.1}$$

Our first technical estimate describes how close $F(r)$ is to r, for large r.

Lemma 3.1 *Let $\kappa > 0$ and $0 < \delta_0 < 1/2$. Define*

$$E_1(r) := \left\{ \left(1 - \frac{5}{\kappa} r^{-\delta_0}\right) r \le F(r) \le \left(1 + \frac{5}{\kappa} r^{-\delta_0}\right) r \right\}. \tag{3.2}$$

Then for all large r,

$$\mathbb{P}(E_1(r)^c) \le \exp(-r^{1-2\delta_0}/4). \tag{3.3}$$

As a consequence, for any $\varepsilon > 0$, we have, almost surely, for all large r,

$$(1-\varepsilon) r \le F(r) \le (1+\varepsilon) r. \tag{3.4}$$

With an abuse of notation, for $r \ge 0$, we denote by $X \circ \Theta_{H(r)}$ the process $(X(H(r)+t) - r,\ t \ge 0)$, which, conditionally on W_κ, is a diffusion in the potential $(W_\kappa(x+r) - W_\kappa(r),\ x \in \mathbb{R})$, starting from 0. Define $H_{X\circ\Theta_{H(r)}}(s) = H(r+s) - H(r)$. Similarly, $F_{X\circ\Theta_{H(r)}}$, $L^*_{X\circ\Theta_{H(r)}}$, $(H \circ F)_{X\circ\Theta_{H(r)}}$, etc, denote respectively the processes F, L^* and $H \circ F$ for the diffusion $X \circ \Theta_{H(r)}$, with $(L^*)_X := L^*_X$. The following lemma is a modification of the Borel–Cantelli lemma.

Lemma 3.2 *Let $\kappa > 0$. Let $f : (0, +\infty)^2 \to \mathbb{R}$ be a continuous function, and let $(\Delta_n)_{n\ge1}$ be a sequence of open sets in $\mathbb{R}$. Let $\alpha > 0$, $r_n := \exp(n^\alpha)$ and $R_n := \sum_{k=1}^n r_k$. If*

$$\sum_{n\ge1} \mathbb{P}\left\{ f[(H\circ F)(r_{2n}), (L^*_X \circ H \circ F)(r_{2n})] \in \Delta_n \right\} = +\infty, \tag{3.5}$$

then for any $\varepsilon > 0$, almost surely, there exist infinitely many n such that for some $t_n \in [(1-\varepsilon)r_{2n}, (1+\varepsilon)r_{2n}]$,

$$f[H_{X\circ\Theta_{H(R_{2n-1})}}(t_n), (L^* \circ H)_{X\circ\Theta_{H(R_{2n-1})}}(t_n)] \in \Delta_n.$$

In the rest of the paper, we define, for $\delta_1 > 0$,

$$\lambda := 4(1+\kappa), \quad c_3 := 2(\lambda/\kappa)^{\delta_1}, \quad \psi_\pm(r) := 1 \pm \frac{c_3}{r^{\delta_1}}, \quad t_\pm(r) := \frac{\kappa \psi_\pm(r) r}{\lambda}. \tag{3.6}$$

Moreover, if $(\beta(s),\ s\geq 0)$ is a Brownian motion and $v>0$, we define the Brownian motion $(\beta_v(s),\ s\geq 0)$ by $\beta_v(s):=(1/v)\beta(v^2 s)$. We also introduce

$$
\begin{aligned}
K_\beta(\kappa) &:= \int_0^{+\infty} x^{1/\kappa-2} L_\beta(\tau_\beta(\lambda),x)\mathrm{d}x, \qquad 0<\kappa<1, && (3.7)\\
C_\beta &:= \int_0^1 \frac{L_\beta(\tau_\beta(8),x)-8}{x}\mathrm{d}x + \int_1^{+\infty}\frac{L_\beta(\tau_\beta(8),x)}{x}\mathrm{d}x. && (3.8)
\end{aligned}
$$

We prove in Section 7 the following approximation.

Lemma 3.3 *Let $\varepsilon\in(0,1)$. For $\delta_1>0$ small enough, there exist a Brownian motion $(\beta(t),\ t\geq 0)$ and a constant $c_4>0$ such that the following holds:*
(i) *For $\kappa>0$, some $\alpha>0$ and all large r, we have*

$$\mathbb{P}\{E_2(r)\}\geq 1-r^{-\alpha},$$

where

$$
\begin{aligned}
E_2(r) &:= \left\{(1-\varepsilon)\widehat{L}_-(r)\leq L_X^*[H(F(r))]\leq(1+\varepsilon)\widehat{L}_+(r)\right\}, && (3.9)\\
\widehat{L}_\pm(r) &:= 4[\kappa t_\pm(r)]^{1/\kappa}\sup_{0\leq u\leq\tau_{\beta_{t_\pm(r)}}(\lambda)}[\beta_{t_\pm(r)}(u)]^{1/\kappa} = 4\sup_{0\leq u\leq \tau_\beta(\lambda t_\pm(r))}[\kappa\beta(u)]^{1/\kappa}. && (3.10)
\end{aligned}
$$

(ii) *For $0<\kappa\leq 1$, some $\alpha>0$ and all large r, we have*

$$\mathbb{P}\{E_3(r)\}\geq 1-r^{-\alpha},$$

where

$$
\begin{aligned}
E_3(r) &:= \left\{(1-\varepsilon)\widehat{I}_-(r)\leq H(F(r))\leq(1+\varepsilon)\widehat{I}_+(r)\right\}, && (3.11)\\
\widehat{I}_\pm(r) &:= \begin{cases} 4\kappa^{1/\kappa-2}t_\pm(r)^{1/\kappa}\{K_{\beta_{t_\pm(r)}}(\kappa)\pm c_4 t_\pm(r)^{1-1/\kappa}\}, & 0<\kappa<1,\\ 4t_\pm(r)\{C_{\beta_{t_\pm(r)}}+8\log t_\pm(r)\}, & \kappa=1.\end{cases} && (3.12)
\end{aligned}
$$

The rest of this section is devoted to the proofs of Lemmas 3.1 and 3.2. The proof of Lemma 3.3 is postponed to Section 7.

Proof of Lemma 3.1. Let $0<\delta_0<1/2$, and fix $r>0$. Define $\widetilde{W}_\kappa(u):=W(u+r)-W(r)-\kappa u/2$, and $\widetilde{A}_\infty:=\int_0^\infty\exp(\widetilde{W}_\kappa(u))\mathrm{d}u$. Notice that

$$\log[A_\infty-A(r)]=\log\widetilde{A}_\infty+W_\kappa(r).$$

Define

$$E_4(r):=\left\{\left(-2r^{-\delta_0}-\frac{\kappa}{2}\right)r\leq\log[A_\infty-A(r)]\leq\left(2r^{-\delta_0}-\frac{\kappa}{2}\right)r\right\}.$$

Recall that $\widetilde{A}_\infty \stackrel{\mathcal{L}}{=} 2/\gamma_\kappa$, where γ_κ is a gamma variable of parameter κ (see e.g. Dufresnes, [9]), i.e., γ_κ has density $\frac{1}{\Gamma(\kappa)} e^{-x} x^{\kappa-1}$ for positive x. Consequently,

$$\begin{aligned} \mathbb{P}(E_4(r)^c) &= \mathbb{P}[|\log(A_\infty - A(r)) + \kappa r/2| > 2r^{1-\delta_0}] \\ &\leq \mathbb{P}[\gamma_\kappa > 2\exp(r^{1-\delta_0})] + \mathbb{P}[\gamma_\kappa < 2\exp(-r^{1-\delta_0})] + \mathbb{P}[|W(r)| > r^{1-\delta_0}] \\ &\leq 3\exp[-r^{1-2\delta_0}/2] \end{aligned} \tag{3.13}$$

for r large enough. Recall that $A_\infty - A(F(r)) = \delta(r) = e^{-\kappa r/2}$. On $E_4[(1+5r^{-\delta_0}/\kappa)r]$,

$$\begin{aligned} \log\{A_\infty - A[(1+5r^{-\delta_0}/\kappa)r]\} &\leq \{2[(1+5r^{-\delta_0}/\kappa)r]^{-\delta_0} - \kappa/2\}(1+5r^{-\delta_0}/\kappa)r \\ &= \frac{\kappa}{2} r \left(-1 + \frac{4}{\kappa} r^{-\delta_0} + o(r^{-\delta_0})\right)(1 + \frac{5r^{-\delta_0}}{\kappa}) \\ &< \log[A_\infty - A(F(r))], \end{aligned}$$

where $f(r) = o(g(r))$ means $\lim_{r\to 0} \frac{f(r)}{g(r)} = 0$. This gives the second inequality in (3.2) by monotonicity of A. A similar argument shows that the first inequality holds on $E_4[(1 - 5r^{-\delta_0}/\kappa)r]$. This yields (3.3) in view of (3.13).

Inequality (3.4) follows from (3.3) by an application of the Borel–Cantelli lemma and the monotonicity of $F(\cdot)$. □

Proof of Lemma 3.2. We divide $\mathbb{R}_+$ into some regions in which the diffusion X will behave "independently", in order to apply the Borel–Cantelli lemma.

Let $n \geq 1$ and let

$$E_5(n) := \left\{ \inf_{t:\, H(R_{2n-1}) \leq t \leq H(R_{2n} + r_{2n+1}/2)} X(t) > R_{2n-2} + \frac{1}{2} r_{2n-1} \right\}.$$

Define $x_n := r_{2n-1}/2$. First, we notice that for any environment, i.e., for any realization of W_κ,

$$\begin{aligned} P_\omega(E_5(n)^c) &= P_\omega^{R_{2n-1}}[H(R_{2n-2} + x_n) < H(R_{2n} + x_{n+1})] \\ &= \left(1 + \frac{\int_{R_{2n-2}+x_n}^{R_{2n-1}} e^{W_\kappa(x)} \mathrm{d}x}{\int_{R_{2n-1}}^{R_{2n}+x_{n+1}} e^{W_\kappa(x)} \mathrm{d}x}\right)^{-1}. \end{aligned}$$

Let $\varepsilon_0 > 0$, $v_n := 2(\log n)/\kappa$ and

$$\begin{aligned} E_6 &:= \left\{ \sup_{0 \leq x \leq r_{2n-1} - x_n} \left| W_\kappa(x + R_{2n-2} + x_n) - W_\kappa(R_{2n-2} + x_n) + \frac{\kappa}{2} x \right| \leq \varepsilon_0 (r_{2n-1} - x_n) \right\}, \\ E_7 &:= \left\{ \sup_{x \geq 0} [W_\kappa(x + R_{2n-1}) - W_\kappa(R_{2n-1})] \leq v_n \right\}. \end{aligned}$$

For large n,

$$\mathbb{P}(E_6^c) \leq 2\exp[-\varepsilon_0^2 (r_{2n-1} - x_n)/2]. \tag{3.14}$$

Moreover (see Borodin and Salminen [5], formula 1.1.4 (1)),

$$\mathbb{P}(E_7^c) = \exp(-\kappa v_n) = 1/n^2. \tag{3.15}$$

On the other hand, we have, on $E_6 \cap E_7$,

$$\int_{R_{2n-1}}^{R_{2n}+x_{n+1}} e^{W_\kappa(x)} \mathrm{d}x \leq (r_{2n} + x_{n+1}) \exp\left(v_n + W_\kappa(R_{2n-1})\right),$$

and for n large enough (on $E_6 \cap E_7$),

$$\begin{aligned}\int_{R_{2n-2}+x_n}^{R_{2n-1}} e^{W_\kappa(x)} \mathrm{d}x &\geq \int_0^{r_{2n-1}-x_n} e^{W_\kappa(R_{2n-2}+x_n) - \frac{\kappa}{2}x - \varepsilon_0(r_{2n-1}-x_n)} \mathrm{d}x \\ &\geq \exp[W_\kappa(R_{2n-2}+x_n) - \varepsilon_0(r_{2n-1}-x_n)]/\kappa.\end{aligned}$$

As a consequence, on $E_6 \cap E_7$,

$$\begin{aligned}P_\omega(E_5(n)^c) &\leq \kappa \frac{(r_{2n}+x_{n+1}) \exp[v_n + W_\kappa(R_{2n-1})]}{\exp[W_\kappa(R_{2n-2}+x_n) - \varepsilon_0(r_{2n-1}-x_n)]} \\ &\leq \kappa(r_{2n}+x_{n+1}) \exp[v_n + (2\varepsilon_0 - \kappa/2)(r_{2n-1}-x_n)]. \end{aligned} \tag{3.16}$$

Integrating (3.16) over $E_6 \cap E_7$, and recalling (3.14) and (3.15), we get for ε_0 small enough,

$$\sum_{n=1}^{+\infty} \mathbb{P}(E_5(n)^c) < \infty. \tag{3.17}$$

To complete the proof of Lemma 3.2, we define

$$\begin{aligned}\mathcal{D}_n &:= \Big\{ \exists t_n \in [(1-\varepsilon)r_{2n}, (1+\varepsilon)r_{2n}], \\ &\qquad f[H_{X\circ\Theta_{H(R_{2n-1})}}(t_n), (L^*\circ H)_{X\circ\Theta_{H(R_{2n-1})}}(t_n)] \in \Delta_n \Big\}, \\ \mathcal{E}_n &:= \left\{ \left(1 - \frac{5}{\kappa} r_{2n}^{-\delta_0}\right) r_{2n} \leq F_{X\circ\Theta_{H(R_{2n-1})}}(r_{2n}) \leq \left(1 + \frac{5}{\kappa} r_{2n}^{-\delta_0}\right) r_{2n} \right\}.\end{aligned}$$

Let $\widetilde{t}_n := F_{X\circ\Theta_{H(R_{2n-1})}}(r_{2n})$. We have,

$$\mathcal{D}_n \cap E_5(n) \supset \Big\{ f[H_{X\circ\Theta_{H(R_{2n-1})}}(\widetilde{t}_n), (L^*\circ H)_{X\circ\Theta_{H(R_{2n-1})}}(\widetilde{t}_n)] \in \Delta_n \Big\} \cap E_5(n) \cap \mathcal{E}_n.$$

By assumption, $\sum_n \mathbb{P}\{f[H_{X\circ\Theta_{H(R_{2n-1})}}(\widetilde{t}_n), (L^*\circ H)_{X\circ\Theta_{H(R_{2n-1})}}(\widetilde{t}_n)] \in \Delta_n\} = \infty$. Moreover, $X\circ\Theta_{H(R_{2n-1})}$ is a diffusion process in the potential $W_\kappa(x+R_{2n-1}) - W_\kappa(R_{2n-1})$, $x \in \mathbb{R}$, hence $\mathbb{P}(\mathcal{E}_n) = \mathbb{P}(E_1(r_{2n}))$. In view of (3.17) and Lemma 3.1, this yields $\sum_{n\in\mathbb{N}} \mathbb{P}(\mathcal{D}_n \cap E_5(n)) = +\infty$.

Since $\mathcal{D}_n \cap E_5(n)$, $n \geq 1$, are independent events, Lemma 3.2 follows by an application of the Borel–Cantelli lemma. $\square$

The proof of Lemma 3.3 is postponed to Section 7.

4 Proof of Theorems 1.2 and 1.3

4.1 Proof of Theorem 1.3

Let $r_n := e^n$ and $R_n := \sum_{k=1}^n r_k$. Let $a(\cdot)$ be a positive nondecreasing function. We begin with the upper bound in Theorem 1.3.

According to Formula 4.1.2 of Borodin and Salminen [5],

$$\mathbb{P}\left(\sup_{0 \le t \le \tau_\beta(v)} \beta(t) < y \right) = \exp\left(-\frac{v}{2y} \right), \qquad v > 0,\ y > 0. \tag{4.1}$$

In particular, for $\widehat{L}_\pm$ which is defined in (3.10), and any positive y and r,

$$\begin{aligned} \mathbb{P}\left(\widehat{L}_\pm(r) < (yr)^{1/\kappa} \right) &= \mathbb{P}\left[\sup_{0 \le u \le \tau_{\beta_{t_\pm(r)}}(\lambda)} \beta_{t_\pm(r)}(u) < \frac{yr}{4^\kappa \kappa t_\pm(r)} \right] \\ &= \exp\left(-\frac{\kappa^2 4^\kappa \psi_\pm(r)}{2y} \right). \end{aligned} \tag{4.2}$$

This together with Lemma 3.3 gives, for r large enough,

$$\begin{aligned} \mathbb{P}\left\{ L_X^*[H(F(r))] > \left(ra(e^{-2}r)\right)^{1/\kappa} \right\} &\le \mathbb{P}\left\{ (1+\varepsilon)\widehat{L}_+(r) > \left(ra(e^{-2}r)\right)^{1/\kappa} \right\} + \mathbb{P}(E_2(r)^c) \\ &\le 1 - \exp\left(-\frac{(1+\varepsilon)^\kappa \kappa^2 4^\kappa \psi_+(r)}{2a(e^{-2}r)} \right) + \frac{1}{(\log r)^2} \\ &\le \frac{c_5}{a(e^{-2}r)} + \frac{1}{(\log r)^2} \end{aligned} \tag{4.3}$$

since $1 - e^{-x} \le x$ for all $x \in \mathbb{R}$.

Assume $\sum_{n=1}^{+\infty} \frac{1}{na(n)} < \infty$, which is equivalent to $\sum_{n=1}^{+\infty} \frac{1}{a(r_n)} < \infty$. Then

$$\sum_{n=1}^{+\infty} \mathbb{P}\{L_X^*[H(F(r_n))] > [r_n a(r_{n-2})]^{1/\kappa}\} < \infty.$$

By the Borel–Cantelli lemma, almost surely for all large n, $L_X^*[H(F(r_n))] \le [r_n a(r_{n-2})]^{1/\kappa}$. On the other hand, $r_{n-1} \le F(r_n)$ almost surely for all large n (see (3.4)). As a consequence, almost surely for all large n, $L_X^*[H(r_{n-1})] \le [r_n a(r_{n-2})]^{1/\kappa}$.

Let $r \in [r_{n-2}, r_{n-1}]$. Then

$$L_X^*[H(r)] \le L_X^*[H(r_{n-1})] \le [r_n a(r_{n-2})]^{1/\kappa} \le e^{2/\kappa}[ra(r)]^{1/\kappa}.$$

Consequently,

$$\limsup_{r \to +\infty} \frac{L_X^*[H(r)]}{[ra(r)]^{1/\kappa}} \le e^{2/\kappa} \qquad \mathbb{P}\text{–a.s.} \tag{4.4}$$

Since $\sum_{n=1}^{+\infty} \frac{1}{\varepsilon a(e^n)}$ is also finite, (4.4) holds for $a(\cdot)$ replaced by $\varepsilon a(\cdot)$. Letting $\varepsilon \to 0$ yields the "zero" part of Theorem 1.3.

Now we turn to the proof of the lower bound. Assume $\sum_{n=1}^{+\infty} \frac{1}{a(r_n)} = +\infty$. Observe that we may restrict ourselves to the case $a(x) \to +\infty$ when $x \to +\infty$, since the result in this case yields the result when a is bounded.

By an argument similar to the one leading to (4.3), we have, for r large enough,

$$\mathbb{P}\left\{L_X^*[H(F(r))] > \left(ra(e^2 r)\right)^{1/\kappa}\right\} \geq \frac{c_5}{2a(e^2 r)} - \frac{1}{(\log r)^2},$$

which implies

$$\sum_{n=1}^{+\infty} \mathbb{P}\left\{(L_X^* \circ H \circ F)(r_{2n}) > [r_{2n} a(r_{2n+2})]^{1/\kappa}\right\} = +\infty.$$

By Lemma 3.2, almost surely, there exist infinitely many n such that

$$\sup_{t\in[(1-\varepsilon)r_{2n},(1+\varepsilon)r_{2n}]} (L^* \circ H)_{X\circ\Theta_{H(R_{2n-1})}}(t) > [r_{2n} a(r_{2n+2})]^{1/\kappa}.$$

For such n, we have $\sup_{t\in[(1-\varepsilon)r_{2n},(1+\varepsilon)r_{2n}]} L_X^*[H(R_{2n-1}+t)] > [r_{2n} a(r_{2n+2})]^{1/\kappa}$. Consequently,

$$\sup_{t\in[(1-\varepsilon)r_{2n},(1+\varepsilon)r_{2n}]} \frac{L_X^*(H(R_{2n-1}+t))}{[(R_{2n-1}+t)a(R_{2n-1}+t)]^{1/\kappa}} \geq c_6,$$

almost surely for infinitely many n, which gives

$$\limsup_{r\to+\infty} \frac{L_X^*(H(r))}{[ra(r)]^{1/\kappa}} \geq c_6 \qquad \mathbb{P}\text{–a.s.}$$

Replace $a(\cdot)$ by $a(\cdot)/\varepsilon$, and let $\varepsilon \to 0$. This yields the "infinity" part of Theorem 1.3. $\square$

4.2 Proof of Theorem 1.2

By Lemma 3.3 and (4.2), for every positive function g and large r,

$$\mathbb{P}\left[L_X^*[H(F(r))] < \left(\frac{r}{g(r)}\right)^{1/\kappa}\right] \leq \exp\left(-\frac{\kappa^2 4^\kappa (1-\varepsilon)^\kappa \psi_-(r) g(r)}{2}\right) + \frac{1}{(\log r)^2}. \qquad (4.5)$$

We choose $g(r) := \frac{2(1+\varepsilon)}{\kappa^2 4^\kappa (1-\varepsilon)^{\kappa+1}\psi_-(r)} \log\log r$. Let $s_n := \exp(n^{1-\varepsilon})$. Then

$$\sum_{n=1}^{\infty} \mathbb{P}\left[L_X^*[H(F(s_n))] < \left(\frac{s_n}{g(s_n)}\right)^{1/\kappa}\right] < \infty.$$

By the Borel–Cantelli lemma, almost surely for all large n,

$$L_X^*[H(F(s_n))] \geq [s_n/g(s_n)]^{1/\kappa}.$$

On the other hand, by Lemma 3.1, $s_n \geq F(s_{n-1})$ almost surely for all large n, which implies that, for $r \in [s_n, s_{n+1}]$,

$$L_X^*[H(r)] \geq L_X^*[H(F(s_{n-1}))] \geq [s_{n-1}/g(s_{n-1})]^{1/\kappa} \geq (1-\varepsilon)[r/g(r)]^{1/\kappa},$$

since $s_{n-1}/s_{n+1} \to 1$ as $n \to +\infty$. Consequently,

$$\liminf_{r\to\infty} \frac{L_X^*(H(r))}{(r/\log\log r)^{1/\kappa}} \geq 4\left(\frac{\kappa^2}{2}\right)^{1/\kappa} \qquad \mathbb{P}\text{–a.s.}$$

Now we prove the inequality "$\leq$". Let $r_n := \exp(n^{1+\varepsilon})$, $R_n := \sum_{k=1}^n r_k$ and $\widetilde{g}(r) := \frac{2(1-\varepsilon)}{\kappa^2 4^\kappa (1+\varepsilon)^{\kappa+1}\psi_+(r)} \log\log r$. By Lemma 3.3 and (4.2), for all large r,

$$\mathbb{P}\left[L_X^*[H(F(r))] < \left(\frac{r}{\widetilde{g}(r)}\right)^{1/\kappa}\right] \geq \exp\left(-\frac{\kappa^2 4^\kappa (1+\varepsilon)^\kappa \psi_+(r)\widetilde{g}(r)}{2}\right) - \frac{1}{(\log r)^2}.$$

Therefore,

$$\sum_{n\geq 1} \mathbb{P}\left[L_X^*[H(F(r_{2n}))] < \left(\frac{r_{2n}}{\widetilde{g}(r_{2n})}\right)^{1/\kappa}\right] = +\infty.$$

It follows from Lemma 3.2 that, almost surely, there are infinitely many n such that

$$\inf_{t\in[(1-\varepsilon)r_{2n},(1+\varepsilon)r_{2n}]} (L^* \circ H)_{X\circ\Theta_{H(R_{2n-1})}}(t) < \left(\frac{r_{2n}}{\widetilde{g}(r_{2n})}\right)^{1/\kappa}.$$

On the other hand, an application of Theorem 1.3 gives that almost surely for large n,

$$L_X^*(H(R_{2n-1})) \leq [R_{2n-1}\log^2 R_{2n-1}]^{1/\kappa} \leq \varepsilon\left(\frac{r_{2n}}{\widetilde{g}(r_{2n})}\right)^{1/\kappa},$$

since $R_p \leq p r_p \leq p\exp(-p^\varepsilon) r_{p+1}$ for p large enough. Therefore, almost surely, for infinitely many n,

$$\inf_{t\in[(1-\varepsilon)r_{2n},(1+\varepsilon)r_{2n}]} L_X^*[H(R_{2n-1}+t)] \leq (1+\varepsilon)\left(\frac{r_{2n}}{\widetilde{g}(r_{2n})}\right)^{1/\kappa}.$$

Hence, for such n,

$$\inf_{t\in[(1-\varepsilon)r_{2n},(1+\varepsilon)r_{2n}]} \frac{L_X^*[H(R_{2n-1}+t)]}{[(R_{2n-1}+t)/\log\log(R_{2n-1}+t)]^{1/\kappa}} \leq (1+c_7\varepsilon)\left(\frac{\kappa^2 4^\kappa \psi_+(r_{2n})}{2}\right)^{1/\kappa}.$$

This yields

$$\liminf_{r\to+\infty} \frac{L_X^*(H(r))}{(r/\log\log r)^{1/\kappa}} \leq 4\left(\frac{\kappa^2}{2}\right)^{1/\kappa} \qquad \mathbb{P}\text{–a.s.},$$

proving Theorem 1.2. □

5 Proof of Theorems 1.8 and 1.9

In this section, we assume $0 < \kappa \leq 1$, and prove Theorems 1.8 and 1.9 in Subsections 5.1 and 5.2 respectively.

Let S_κ^{ca} be a (positive) completely asymmetric stable variable of index κ, and C_8^{ca} a completely asymmetric Cauchy variable of parameter 8. Their characteristic functions are given by:

$$\begin{aligned}
\mathbb{E}\exp(itS_\kappa^{ca}) &= \exp\left[-|t|^\kappa\left(1 - i\,\mathrm{sgn}(t)\tan(\frac{\pi\kappa}{2})\right)\right], \qquad (5.1)\\
\mathbb{E}\exp(itC_8^{ca}) &= \exp\left[-8\left(|t| + it\frac{2}{\pi}\log|t|\right)\right].
\end{aligned}$$

We write

$$\psi(\kappa) := \left(\frac{\pi\kappa}{4\Gamma^2(\kappa)\sin(\pi\kappa/2)}\right)^{1/\kappa}. \qquad (5.2)$$

Recall $\widehat{I}_\pm$ from (3.12). By Biane and Yor [4], for $\lambda > 0$ and $0 < \kappa < 1$,

$$\begin{aligned}
\widehat{I}_\pm(r) &\stackrel{\mathcal{L}}{=} 4\kappa^{1/\kappa-2}t_\pm(r)^{1/\kappa}\{2\kappa^{2-2/\kappa}\psi(\kappa)\lambda^{1/\kappa}S_\kappa^{ca} \pm c_4 t_\pm(r)^{1-1/\kappa}\}\\
&= t_\pm(r)^{1/\kappa}\{c_8\, S_\kappa^{ca} \pm c_9\, t_\pm(r)^{1-1/\kappa}\}, \qquad (5.3)
\end{aligned}$$

where

$$c_8 := 8\psi(\kappa)\lambda^{1/\kappa}\kappa^{-1/\kappa}, \qquad c_9 := 4\kappa^{1/\kappa-2}c_4.$$

In case $\kappa = 1$, the result of Biane and Yor [4] says that for some positive constant c_{10},

$$\widehat{I}_\pm(r) \stackrel{\mathcal{L}}{=} 4t_\pm(r)[8c_{10} + (\pi/2)C_8^{ca} + 8\log t_\pm(r)]. \qquad (5.4)$$

We have now all the ingredients to prove Theorems 1.8 and 1.9.

5.1 Proof of Theorem 1.8

5.1.1 Case $0 < \kappa < 1$

We assume $0 < \kappa < 1$. As in Section 4.1, we define $r_n := e^n$ and $R_n := \sum_{k=1}^n r_k$. Let $a(\cdot)$ be a positive nondecreasing function.

By Samorodnitsky and Taqqu ([21], p. 16),

$$\mathbb{P}(S_\kappa^{ca} > x) \underset{x\to+\infty}{\sim} \frac{c_{11}}{x^\kappa},$$

where $f(x) \sim_{x\to+\infty} g(x)$ means $\lim_{x\to+\infty} f(x)/g(x) = 1$, and c_{11} is a positive constant depending on κ.

Without loss of generality, we assume that $a(n) \to \infty$ (for $n \to \infty$).

Recall $t_\pm(\cdot)$ from (3.6). By Lemma 3.3 and (5.3), for large r, we have

$$\begin{aligned}
\mathbb{P}[H(F(r)) > (a(e^{-2}r)t_+(r))^{1/\kappa}] &\le \mathbb{P}[(1+\varepsilon)\widehat{I}_+(r) > (a(e^{-2}r)t_+(r))^{1/\kappa}] + \mathbb{P}[E_3(r)^c] \\
&\le \mathbb{P}\left[c_8 S_\kappa^{ca} + c_9 t_+(r)^{1-1/\kappa} > \frac{(a(e^{-2}r))^{1/\kappa}}{1+\varepsilon}\right] + \frac{1}{(\log r)^2} \\
&\le \frac{c_{12}}{a(e^{-2}r)} + \frac{1}{(\log r)^2}. \qquad (5.5)
\end{aligned}$$

Assume $\sum_{n\ge 1} \frac{1}{a(r_n)} < \infty$. By the Borel–Cantelli lemma, almost surely for n large enough,

$$H(F(r_n)) \le [a(r_{n-2})t_+(r_n)]^{1/\kappa} = (\psi_+(r_n)\kappa r_n a(r_{n-2})/\lambda)^{1/\kappa}.$$

On the other hand, by Lemma 3.1, almost surely for all large n, we have $r_{n+1} \le F(r_{n+2})$, which implies that for $r \in [r_n, r_{n+1}]$,

$$H(r) \le H[F(r_{n+2})] \le [\psi_+(r_{n+2})\kappa r_{n+2} a(r_n)/\lambda]^{1/\kappa} \le c_{13}[ra(r)]^{1/\kappa}.$$

Therefore,

$$\limsup_{r\to+\infty} \frac{H(r)}{[ra(r)]^{1/\kappa}} \le c_{13} \qquad \mathbb{P}\text{–a.s.},$$

implying the “zero” part of Theorem 1.8, since we can replace $a(\cdot)$ by any constant multiple of $a(\cdot)$.

To prove the “infinity" part, we assume $\sum_{n\ge 1} \frac{1}{na(n)} = +\infty$, and observe that, by a similar argument leading to (5.5), we have, for r large enough,

$$\mathbb{P}[H(F(r)) > (a(e^2 r)t_-(r))^{1/\kappa}] \ge \frac{c_{14}}{a(e^2 r)} - \frac{1}{(\log r)^2}.$$

It follows from Lemma 3.2 that, almost surely, there exist infinitely many n such that

$$\sup_{t\in[(1-\varepsilon)r_{2n},(1+\varepsilon)r_{2n}]} H_{X\circ\Theta_{H(R_{2n-1})}}(t) > [a(r_{2n+2})t_-(r_{2n})]^{1/\kappa},$$

which implies, for these n,

$$\sup_{t\in[(1-\varepsilon)r_{2n},(1+\varepsilon)r_{2n}]} \frac{H(R_{2n-1}+t)}{[a(R_{2n-1}+t)(R_{2n-1}+t)]^{1/\kappa}} \ge c_{15}.$$

This gives

$$\limsup_{r\to+\infty} \frac{H(r)}{[ra(r)]^{1/\kappa}} \ge c_{15} \qquad \mathbb{P}\text{–a.s.},$$

proving the “infinity” part in Theorem 1.8, in the case $0 < \kappa < 1$. $\square$

5.1.2 Case $\kappa = 1$

Let $r_n := e^n$ and $R_n := \sum_{k=1}^n r_k$. We recall that there exists a constant $c_{16} > 0$ such that $\mathbb{P}(C_8^{ca} > x) \underset{x\to+\infty}{\sim} \frac{c_{16}}{x}$ (see e.g. Samorodnitsky and Taqqu [21], p. 16).

By Lemma 3.3 and (5.4), for large r,

$$\begin{aligned} & \mathbb{P}\left\{H(F(r)) > 4t_+(r)(1+\varepsilon)[8c_{10} + a(e^{-2}r) + 8\log t_+(r)]\right\} \\ \leq\ & \mathbb{P}\left\{\widehat{I}_+(r) > 4t_+(r)[8c_{10} + a(e^{-2}r) + 8\log t_+(r)]\right\} + \mathbb{P}(E_3(r)^c) \\ \leq\ & \frac{2c_{16}}{(2/\pi)a(e^{-2}r)} + \frac{1}{(\log r)^2}. \end{aligned} \tag{5.6}$$

Assume $\sum_{n\geq 1}\frac{1}{na(n)} < \infty$. Then by the Borel–Cantelli lemma, almost surely, for all large n,

$$H(F(r_n)) \leq \frac{(1+\varepsilon)\psi_+(r_n)\kappa r_n}{2}[8c_{10} + a(r_{n-2}) + 8\log(\psi_+(r_n)\kappa r_n/8)],$$

since $t_+(r) = \psi_+(r)\kappa r/8$ (in case $\kappa = 1$).

Under the additional assumption $\limsup_{r\to+\infty}(\log r)/a(r) < \infty$, we have, almost surely, for all large n and $r \in [r_n, r_{n+1}]$ (thus $r \leq F(r_{n+2})$ by Lemma 3.1),

$$H(r) \leq H(F(r_{n+2})) \leq c_{17} r_{n+2}[a(r_n) + \log r_{n+2}] \leq c_{18} r a(r).$$

This yields the “zero” part of Theorem 1.8 in the case $\kappa = 1$.

For the “infinity" part, we assume $\sum_{n\geq 1}\frac{1}{na(n)} = +\infty$. As in (5.6), we have, for large r,

$$\mathbb{P}\left\{H(F(r)) > 4t_-(r)(1-\varepsilon)a(e^2 r)\right\} \geq \frac{c_{16}}{(4/\pi)a(e^2 r)} - \frac{1}{(\log r)^2},$$

which implies

$$\sum_{n\geq 1}\mathbb{P}\left\{H(F(r_{2n})) > 4t_-(r_{2n})(1-\varepsilon)a(r_{2n+2})\right\} = +\infty.$$

By Lemma 3.2, almost surely, there are infinitely many n such that

$$\sup_{u\in[(1-\varepsilon)r_{2n},(1+\varepsilon)r_{2n}]} H_{X\circ\Theta_{H(R_{2n-1})}}(u) > 4t_-(r_{2n})(1-\varepsilon)a(r_{2n+2}),$$

which yields, for such n,

$$\sup_{v\in[R_{2n-1}+(1-\varepsilon)r_{2n},R_{2n-1}+(1+\varepsilon)r_{2n}]}\frac{H(v)}{va(v)} \geq c_{19}.$$

This proves the “infinity” part of Theorem 1.8 in the case $\kappa = 1$. ∎

5.2 Proof of Theorem 1.9

5.2.1 Case $0 < \kappa < 1$

Recall (see Bertoin [3], p. 221) that

$$\log \mathbb{P}(S_\kappa^{ca} < x) \underset{x\to 0,\ x>0}{\sim} -c_{20} x^{-\kappa/(1-\kappa)}, \tag{5.7}$$

where c_{20} is a constant depending only on κ. Consequently, for r large enough, by (5.3) and Lemma 3.3, for any (strictly) positive function f,

$$\begin{aligned}\mathbb{P}[H(F(r)) < t_-(r)^{1/\kappa} f(r)] &\leq \mathbb{P}[(1-\varepsilon)\widehat{I}_-(r) < t_-(r)^{1/\kappa} f(r)] + \mathbb{P}(E_3(r)^c)\\ &\leq \mathbb{P}\left\{c_8 S_\kappa^{ca} - c_9 t_-(r)^{1-1/\kappa} < \frac{f(r)}{1-\varepsilon}\right\} + \frac{1}{\log^2 r}\\ &\leq \mathbb{P}\left\{S_\kappa^{ca} < \frac{f(r) + c_{21} r^{1-1/\kappa}}{(1-\varepsilon)c_8}\right\} + \frac{1}{\log^2 r}\\ &\leq \exp\left[-(c_{20}-\varepsilon)\left(\frac{(1-\varepsilon)c_8}{f(r)+c_{21}r^{1-1/\kappa}}\right)^{\kappa/(1-\kappa)}\right] + \frac{1}{\log^2 r}. \end{aligned} \tag{5.8}$$

Let $s_n := \exp(n^{1-\varepsilon})$ and

$$f(r) := \left(\frac{1-\varepsilon}{1+\varepsilon}\right)^{(1-\kappa)/\kappa} \frac{(1-\varepsilon)c_8(c_{20}-\varepsilon)^{(1-\kappa)/\kappa}}{(\log\log r)^{(1-\kappa)/\kappa}} - \frac{c_{21}}{r^{1/\kappa-1}}.$$

Then $\sum_n^{+\infty} \mathbb{P}[H(F(s_n)) < t_-(s_n)^{1/\kappa} f(s_n)] < \infty$, which, by means of the Borel–Cantelli lemma, implies that, almost surely, for all large n,

$$H(F(s_n)) \geq t_-(s_n)^{1/\kappa} f(s_n).$$

Recall from Lemma 3.1 that, almost surely, for all large n, we have $F(s_n) \leq (1+\varepsilon)s_n$. Let r be large. There exists n (large) such that $(1+\varepsilon)s_n \leq r \leq (1+2\varepsilon)s_n$. Then

$$H(r) \geq H(F(s_n)) \geq t_-(s_n)^{1/\kappa} f(s_n) \geq t_-^{1/\kappa}\left(\frac{r}{1+2\varepsilon}\right) f\left(\frac{r}{1+2\varepsilon}\right).$$

Plugging the value of $t_-(\frac{r}{1+2\varepsilon})$ (defined in (3.6)), this yields

$$\liminf_{r\to+\infty} \frac{H(r)}{r^{1/\kappa}(\log\log r)^{(\kappa-1)/\kappa}} \geq 8\psi(\kappa) c_{20}^{(1-\kappa)/\kappa} \qquad \mathbb{P}\text{–a.s.}$$

To prove the upper bound, let $r_n := \exp(n^{1+\varepsilon})$, $R_n := \sum_{k=1}^n r_k$ and

$$g(r) := \varepsilon + \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{(1-\kappa)/\kappa} \frac{(1+\varepsilon)c_8(c_{20}+\varepsilon)^{(1-\kappa)/\kappa}}{(\log\log r)^{(1-\kappa)/\kappa}} + \frac{c_{22}}{r^{1/\kappa-1}}.$$

By means of an argument similar to the one leading to (5.8), we have $\sum_{n\geq 1}\mathbb{P}[H(F(r_n)) < t_+(r_n)^{1/\kappa}g(r_n)] = +\infty$. By Lemma 3.2, almost surely, there exist infinitely many n such that

$$\inf_{u\in[(1-\varepsilon)r_{2n},(1+\varepsilon)r_{2n}]} H_{X\circ\Theta_{H(R_{2n-1})}}(u) < [t_+(r_{2n})]^{1/\kappa}g(r_{2n}).$$

On the other hand, by Theorem 1.8, we have, almost surely, for all large n,

$$H(R_{2n-1}) < \left[R_{2n-1}\log^2 R_{2n-1}\right]^{1/\kappa} \leq \varepsilon[t_+(r_{2n})]^{1/\kappa}g(r_{2n}),$$

since $R_k \leq k\exp(-k^\varepsilon)r_{k+1}$ for all large k. Hence, almost surely, for infinitely many n,

$$\inf_{v\in[R_{2n-1}+(1-\varepsilon)r_{2n},\, R_{2n-1}+(1+\varepsilon)r_{2n}]} H(v) < (1+\varepsilon)[t_+(r_{2n})]^{1/\kappa}g(r_{2n}),$$

which implies

$$\liminf_{r\to+\infty}\frac{H(r)}{r^{1/\kappa}(\log\log r)^{(\kappa-1)/\kappa}} \leq 8\psi(\kappa)c_{20}^{(1-\kappa)/\kappa} \qquad \mathbb{P}\text{–a.s.}$$

This completes the proof of Theorem 1.9 in the case $0<\kappa<1$. The value of $c_2(\kappa)$ (recalling ψ from (5.2)) is given by

$$c_2(\kappa) = 8\psi(\kappa)c_{20}^{(1/\kappa)-1} = 8\left(\frac{\pi\kappa}{4\Gamma^2(\kappa)\sin(\pi\kappa/2)}\right)^{1/\kappa} c_{20}^{(1/\kappa)-1}, \tag{5.9}$$

where c_{20} is defined in (5.7) and depends on κ. □

5.2.2 Case $\kappa = 1$

Assume $\kappa = 1$ (thus $\lambda = 8$). By Samorodnitsky and Taqqu ([21], Proposition 1.2.12), $\mathbb{E}[\exp(-C_8^{ca})] = 1$ (in the notation of [21], C_8^{ca} is distributed as $S_1(8,1,0)$). Hence,

$$\mathbb{P}[C_8^{ca} \leq -\varepsilon\log r] \leq r^{-\varepsilon}\,\mathbb{E}[\exp(-C_8^{ca})] = r^{-\varepsilon}. \tag{5.10}$$

By Lemma 3.3 and (5.4), we have, for all large r,

$$\begin{aligned}
&\mathbb{P}\left\{H(F(r)) \leq 4t_-(r)(1-2\varepsilon)[8c_{10}+8\log t_-(r)]\right\}\\
\leq\ &\mathbb{P}\left(C_8^{ca} \leq -\frac{16}{\pi}\frac{\varepsilon}{1-\varepsilon}\log t_-(r)\right) + \mathbb{P}(E_3(r)^c)\\
\leq\ &\mathbb{P}\left(C_8^{ca} \leq -\varepsilon\log r\right) + \mathbb{P}(E_3(r)^c) \leq r^{-c_{23}}.
\end{aligned}$$

Let $s_n := \exp(n^{1-\varepsilon})$. Thus, by the Borel–Cantelli lemma, almost surely, for all large n,

$$H(F(s_n)) > 32t_-(s_n)(1-2\varepsilon)[c_{10}+\log t_-(s_n)] \geq 4(1-3\varepsilon)s_n\log s_n.$$

In view of (the last part of) Lemma 3.1, this yields

$$\liminf_{r\to+\infty}\frac{H(r)}{r\log r} \geq 4 \qquad \mathbb{P}\text{–a.s.}$$

The inequality "$\leq$" follows immediately from Theorem 1.1 (that $H(r)/(r\log r) \to 4$ in probability). This completes the proof of Theorem 1.9 in the case $\kappa = 1$. □

6 Proof of Theorems 1.4–1.7

Proof of Theorem 1.4: case $\kappa > 1$. Follows from Theorems 1.2 and 1.1. □

Proof of Theorem 1.6. Follows from Theorems 1.3 and 1.1. □

We now assume $0 < \kappa \le 1$, and need to prove Theorems 1.4, 1.5 and 1.7. Unfortunately, there is no almost sure convergence result for $H(r)$ in this case due to strong fluctuations; hence a joint study of $L_X^*(H(r))$ and $H(r)$ is useful. In Section 6.1, we prove a lemma which will be needed later on. Section 6.2 is devoted to the proof of Theorems 1.4, 1.5 and 1.7 in the case $0 < \kappa < 1$, whereas Section 6.3 to the proof of Theorems 1.4 and 1.5 in the case $\kappa = 1$.

6.1 A lemma

In this section we assume $0 < \kappa \le 1$. Let $\delta_1 > 0$ and recall the definition of $\widehat{L}_\pm(r)$ from (3.10).

Lemma 6.1 *Let $E_8(r) := \{\widehat{L}_-(r) = \widehat{L}_+(r)\}$. For all $\delta_2 \in (0, \delta_1)$ and all large r, we have*

$$\mathbb{P}[E_8(r)^c] \le r^{-\delta_2}.$$

Proof. Observe that

$$\begin{aligned} 1 \le \left(\frac{\widehat{L}_+(r)}{\widehat{L}_-(r)}\right)^\kappa &\le \max\left(1, \frac{\sup_{0\le u\le \tau_\beta[\psi_+(r)\kappa r]-\tau_\beta[\psi_-(r)\kappa r]} \beta\{u+\tau_\beta[\psi_-(r)\kappa r]\}}{\sup_{0\le u\le \tau_\beta(\psi_-(r)\kappa r)} \beta(u)}\right) \\ &= \max\left(1, \frac{\sup_{0\le u\le \tau_{\widetilde{\beta}}\{[\psi_+(r)-\psi_-(r)]\kappa r\}} \widetilde{\beta}(u)}{\sup_{0\le u\le \tau_\beta(\psi_-(r)\kappa r)} \beta(u)}\right), \end{aligned} \tag{6.1}$$

where $\widetilde{\beta}(u) := \beta(u + \tau_\beta[\psi_-(r)\kappa r])$, $u \ge 0$, is a Brownian motion independent of the random variable $\sup_{0\le u\le \tau_\beta(\psi_-(r)\kappa r)} \beta(u)$. By (4.1) and the trivial inequality $1 - e^{-x} \le x$ (for $x \ge 0$),

$$\mathbb{P}\left(\sup_{0\le u\le \tau_{\widetilde{\beta}}\{[\psi_+(r)-\psi_-(r)]\kappa r\}} \widetilde{\beta}(u) > [\psi_+(r)-\psi_-(r)]\kappa r^{1+\delta_2}\right) \le \frac{1}{2r^{\delta_2}},$$

$$\mathbb{P}\left(\sup_{0\le u\le \tau_\beta(\psi_-(r)\kappa r)} \beta(u) < \frac{\psi_-(r)\kappa r}{4\delta_2 \log r}\right) = e^{-2\delta_2 \log r} = \frac{1}{r^{2\delta_2}}.$$

By definition, $\psi_\pm(r) = 1 \pm c_3 r^{-\delta_1}$ (see (3.6)). Therefore, we have, for large r, with probability greater than $1 - r^{\delta_2}$,

$$\frac{\sup_{0\le u\le \tau_{\widetilde{\beta}}\{[\psi_+(r)-\psi_-(r)]\kappa r\}} \widetilde{\beta}(u)}{\sup_{0\le u\le \tau_\beta(\psi_-(r)\kappa r)} \beta(u)} \le \frac{[\psi_+(r)-\psi_-(r)]\kappa r^{1+\delta_2}}{\psi_-(r)\kappa r/(4\delta_2 \log r)} = \frac{8c_3\delta_2 r^{-(\delta_1-\delta_2)}\log r}{1 - c_3 r^{-\delta_1}} < 1,$$

which, combined with (6.1), yields the lemma. □

6.2 Case $0<\kappa<1$

This section is devoted to the proof of Theorems 1.4, 1.5 and 1.7 in the case $0<\kappa<1$.

For any Brownian motion β, let

$$N_\beta := \frac{\int_0^{+\infty} x^{1/\kappa-2} L_\beta(\tau_\beta(\lambda), x)\mathrm{d}x}{\{\sup_{0\le u\le \tau_\beta(\lambda)} \beta(u)\}^{1/\kappa}},$$

so that in the notation of (3.7), (3.6) and (3.10),

$$N_{\beta_{t_\pm(r)}} = 4[\kappa t_\pm(r)]^{1/\kappa} \frac{K_{\beta_{t_\pm(r)}}(\kappa)}{\widehat{L}_\pm(r)}.$$

On $E_2(r)\cap E_3(r)\cap E_8(r)$ (the events $E_2(r)$ and $E_3(r)$ are defined in Lemma 3.3, whereas $E_8(r)$ in Lemma 6.1), we have, for some constant c_{24} and all large r,

$$\begin{aligned}\frac{H(F(r))}{L_X^*[H(F(r))]} &\ge \frac{4\kappa^{1/\kappa-2}t_-(r)^{1/\kappa}(1-\varepsilon)\{K_{\beta_{t_-(r)}}(\kappa)-c_4 t_-(r)^{1-1/\kappa}\}}{(1+\varepsilon)\widehat{L}_-(r)}\\ &\ge (1-3\varepsilon)\frac{N_{\beta_{t_-(r)}}}{\kappa^2} - c_{24}\frac{t_-(r)}{\widehat{L}_-(r)}. \end{aligned}\tag{6.2}$$

Similarly, on $E_2(r)\cap E_3(r)\cap E_8(r)$, for some constant c_{25} and all large r,

$$\frac{H(F(r))}{L_X^*[H(F(r))]} \le (1+3\varepsilon)\frac{N_{\beta_{t_+(r)}}}{\kappa^2} + c_{25}\frac{t_+(r)}{\widehat{L}_+(r)}. \tag{6.3}$$

Define $E_9(r) := \{\frac{c_{24}t_-(r)}{\widehat{L}_-(r)}\le\varepsilon,\ \frac{c_{25}t_+(r)}{\widehat{L}_+(r)}\le\varepsilon\}$. By (4.2), $\mathbb{P}(E_9(r)^c)\le 1/r^2$ for large r. Thus $\mathbb{P}\{E_2(r)\cap E_3(r)\cap E_8(r)\cap E_9(r)\}\ge 1-r^{-\alpha_1}$ for some $\alpha_1>0$ and all large r. In view of (6.2) and (6.3), we have, for some $\alpha_1>0$ and all large r,

$$\mathbb{P}\left((1-3\varepsilon)\frac{N_{\beta_{t_-(r)}}}{\kappa^2}-\varepsilon \le \frac{H(F(r))}{L_X^*[H(F(r))]} \le (1+3\varepsilon)\frac{N_{\beta_{t_+(r)}}}{\kappa^2}+\varepsilon\right)\ge 1-\frac{1}{r^{\alpha_1}}. \tag{6.4}$$

We now proceed to the study of the law of N_β. By the second Ray–Knight theorem (Fact 2.2), there exists a 0–dimensional Bessel process U, starting from $\sqrt{\lambda}$, such that

$$(L_\beta(\tau_\beta(\lambda),x), x\ge 0) = (U^2(x), x\ge 0). \tag{6.5}$$

Moreover,

$$\sup_{0\le u\le\tau_\beta(\lambda)} \beta(u) = \inf\{x\ge 0,\ U(x)=0\} =: \zeta_U. \tag{6.6}$$

Therefore,

$$N_\beta = \zeta_U^{-1/\kappa} \int_0^{\zeta_U} x^{1/\kappa-2} U^2(x)\mathrm{d}x. \tag{6.7}$$

By Williams' time reversal theorem (Fact 2.4), there exists a 4–dimensional Bessel process R, starting from 0, such that

$$(U(\zeta_U - t),\ t \le \zeta_U) \stackrel{\mathcal{L}}{=} (R(t),\ t \le \gamma_a), \tag{6.8}$$

where $a := \sqrt{\lambda}$ and

$$\gamma_a := \sup\{t \ge 0,\ R(t) = \sqrt{\lambda}\}. \tag{6.9}$$

Therefore,

$$N_\beta \stackrel{\mathcal{L}}{=} \gamma_a^{-1/\kappa} \int_0^{\gamma_a} x^{1/\kappa-2} R^2(\gamma_a - x)\mathrm{d}x = \int_0^1 (1-v)^{1/\kappa-2} \left(\frac{R(\gamma_a v)}{\sqrt{\gamma_a}}\right)^2 \mathrm{d}v.$$

Recall (Yor [29], p. 52) that for any bounded measurable functional G,

$$\mathbb{E}\left[G\left(\frac{R(\gamma_a u)}{\sqrt{\gamma_a}}, u \le 1\right)\right] = \mathbb{E}\left(\frac{2}{R^2(1)} G(R(u), u \le 1)\right). \tag{6.10}$$

In particular, for $x > 0$,

$$\mathbb{P}(N_\beta > x) = \mathbb{E}\left(\frac{2}{R^2(1)} \mathbf{1}_{\{\int_0^1 (1-v)^{1/\kappa-2} R^2(v)\mathrm{d}v > x\}}\right). \tag{6.11}$$

6.2.1 Proof of Theorem 1.5 (case $0 < \kappa < 1$)

Fix $y > 0$. By (6.11),

$$\begin{aligned}\mathbb{P}(N_\beta > y\log\log r) &\le \mathbb{E}\left(\frac{2}{R^2(1)} \mathbf{1}_{\{\int_0^1 (1-v)^{1/\kappa-2} R^2(v)\mathrm{d}v > y\log\log r,\ R^2(1)\le 1\}}\right) \\ &\quad + 2\mathbb{P}\left(\int_0^1 (1-v)^{1/\kappa-2} R^2(v)\mathrm{d}v > y\log\log r\right) \\ &:= \Pi_1(r) + \Pi_2(r) \end{aligned} \tag{6.12}$$

with obvious notation.

We first consider $\Pi_2(r)$. Recall that $\mathcal{C}_0$ denotes the set of continuous functions $\phi : [0,1] \to \mathbb{R}$, such that $\phi(0) = 0$. By Schilder's theorem (see Dembo and Zeitouni [8], p. 185),

$$\begin{aligned} &\lim_{r\to+\infty} \frac{1}{y\log\log r} \log \mathbb{P}\left(\int_0^1 (1-v)^{1/\kappa-2} R^2(v)\mathrm{d}v > y\log\log r\right) \\ &= -\inf\left\{\frac{1}{2}\int_0^1 \phi'(v)^2\mathrm{d}v : \quad \phi \in \mathcal{C}_0,\ \int_0^1 (1-v)^{1/\kappa-2}\phi^2(v)\mathrm{d}v \ge 1\right\} \\ &= -c_1(\kappa). \end{aligned} \tag{6.13}$$

For $\phi \in \mathcal{C}_0$, $\phi^2(v) = (\int_0^v \phi'(u)\mathrm{d}u)^2 \le v\int_0^1 \phi'(u)^2\mathrm{d}u$; thus $\int_0^1 (1-v)^{1/\kappa-2}\phi^2(v)\mathrm{d}v \le [\int_0^1 (1-v)^{1/\kappa-2}v\mathrm{d}v]\int_0^1 \phi'(v)^2\mathrm{d}v$, which implies $c_1(\kappa)>0$.

By (6.13), for large r,

$$\Pi_2(r) \le \frac{1}{(\log r)^{(1-\varepsilon)yc_1(\kappa)}}. \tag{6.14}$$

Now, we consider $\Pi_1(r)$. As R is the Euclidean norm of a 4–dimensional Brownian motion $(\gamma(t),\ t\ge 0)$, we have

$$\Pi_1(r) = \mathbb{E}\left(\frac{2}{\|\gamma(1)\|^2}\mathbf{1}_{\{\|\gamma(1)\|\le 1\}}\mathbf{1}_{\{\int_0^1 (1-v)^{1/\kappa-2}\|\gamma(v)\|^2\mathrm{d}v > y\log\log r\}}\right),$$

where $\|\cdot\|$ denotes the Euclidean norm. By the triangular inequality, for any positive measure μ on $[0,1]$, $\sqrt{\int_0^1 \|\gamma(v)\|^2\mathrm{d}\mu(v)} \le \sqrt{\int_0^1 \|\gamma(v)-v\gamma(1)\|^2\mathrm{d}\mu(v)} + \sqrt{\int_0^1 v^2\mathrm{d}\mu(v)}\,\|\gamma(1)\|$. Therefore,

$$\begin{aligned}\Pi_1(r) &\le \mathbb{E}\left(\frac{2}{\|\gamma(1)\|^2}\mathbf{1}_{\{\int_0^1 (1-v)^{1/\kappa-2}\|\gamma(v)-v\gamma(1)\|^2\mathrm{d}v > (\sqrt{y\log\log r}-c_{26})^2\}}\right)\\ &:= \mathbb{E}\left(\frac{2}{\|\gamma(1)\|^2}\mathbf{1}_E\right).\end{aligned}$$

By the independence of $\gamma(1)$ and $(\gamma(v)-v\gamma(1),\ v\in[0,1])$, the expectation on the right hand side is $=\mathbb{E}(\frac{2}{\|\gamma(1)\|^2})\mathbb{P}(E)=\mathbb{P}(E)$ (the last identity being a consequence of (6.10) by taking $G:=1$ there). Therefore, $\Pi_1(r)\le\mathbb{P}(E)$.
Again, by the independence of $\gamma(1)$ and $(\gamma(v)-v\gamma(1),\ v\in[0,1])$, we see that, by writing $c_{27} := 1/\mathbb{P}(\|\gamma(1)\|\le 1)$, $\Pi_1(r)\le c_{27}\,\mathbb{P}(E,\ \|\gamma(1)\|\le 1)$. By another application of the triangular inequality, this leads to:

$$\Pi_1(r) \le c_{27}\,\mathbb{P}\left(\int_0^1 (1-v)^{1/\kappa-2}\|\gamma(v)\|^2\mathrm{d}v > (\sqrt{y\log\log r}-2c_{26})^2\right).$$

In view of (6.13), we have, for all large r,

$$\Pi_1(r) \le \frac{1}{(\log r)^{(1-\varepsilon)yc_1(\kappa)}}.$$

Plugging this into (6.12) and (6.14) yields that, for any $y>0$, $\varepsilon>0$ and all large r,

$$\mathbb{P}(N_\beta > y\log\log r) \le \frac{2}{(\log r)^{(1-\varepsilon)yc_1(\kappa)}}.$$

Let $s_n := \exp(n^{1-\varepsilon})$. In view of (6.4), we have proved that $\sum_{n=1}^{+\infty}\mathbb{P}\{\frac{H(F(s_n))}{L_X^*[H(F(s_n))]} > \frac{1+4\varepsilon}{(1-\varepsilon)^2\kappa^2c_1(\kappa)}\log\log s_n\}<\infty$. By the Borel–Cantelli lemma, almost surely, for all large n,

$$\frac{H(F(s_n))}{L_X^*[H(F(s_n))]} \le \frac{1+4\varepsilon}{(1-\varepsilon)^2\kappa^2c_1(\kappa)}\log\log s_n. \tag{6.15}$$

We now bound $\frac{H(F(s_{n+1}))}{H(F(s_n))}$. Observe that for large n, $s_{n+1} - s_n \le n^{-\varepsilon} s_n$. By Lemma 3.1, almost surely for all large n,

$$\begin{aligned} H(F(s_{n+1})) - H(F(s_n)) &\le H\left[(1+\frac{5}{\kappa}s_{n+1}^{-\delta_0})s_{n+1}\right] - H\left[(1-\frac{5}{\kappa}s_n^{-\delta_0})s_n\right] \\ &\le H\left[(1-\frac{5}{\kappa}s_n^{-\delta_0})s_n + (2-\varepsilon)n^{-\varepsilon}s_n\right] - H\left[(1-\frac{5}{\kappa}s_n^{-\delta_0})s_n\right] \\ &= \inf\left\{u \ge 0 :\ \widehat{X}_n(u) > (2-\varepsilon)n^{-\varepsilon}s_n\right\}, \end{aligned} \tag{6.16}$$

where $(\widehat{X}_n(u),\ u \ge 0)$ is a diffusion process in the random potential $\widehat{W}_\kappa(x) := W_\kappa(x + (1 - \frac{5}{\kappa}s_n^{-\delta_0})s_n) - W_\kappa((1-\frac{5}{\kappa}s_n^{-\delta_0})s_n)$, $x \in \mathbb{R}$. It is natural to write the above identity as

$$\inf\left\{u \ge 0 :\ \widehat{X}_n(u) > (2-\varepsilon)n^{-\varepsilon}s_n\right\} = \widehat{H}_n[(2-\varepsilon)n^{-\varepsilon}s_n]. \tag{6.17}$$

Note that for any $r > 0$, under $\mathbb{P}$, $\widehat{H}_n(r)$ is distributed as $H(r)$. Therefore, applying (5.5) and Lemma 3.1 to $r = 2n^{-\varepsilon}s_n$ yields that, for any $0 < \delta_0 < \frac{1}{2}$,

$$\sum_n \mathbb{P}\left[\widehat{H}_n\left([1-\frac{5}{\kappa}(2n^{-\varepsilon}s_n)^{-\delta_0}]2n^{-\varepsilon}s_n\right) > [n(\log n)^{1+\varepsilon}t_+(2n^{-\varepsilon}s_n)]^{1/\kappa}\right] < \infty.$$

Since $[1-\frac{5}{\kappa}(2n^{-\varepsilon}s_n)^{-\delta_0}]2n^{-\varepsilon}s_n \ge (2-\varepsilon)n^{-\varepsilon}s_n$ (for large n), it follows from the Borel–Cantelli lemma that, almost surely, for all large n,

$$\widehat{H}_n\left((2-\varepsilon)n^{-\varepsilon}s_n\right) \le [n(\log n)^{1+\varepsilon}t_+(2n^{-\varepsilon}s_n)]^{1/\kappa}. \tag{6.18}$$

This, together with (6.16) and (6.17), yields that, almost surely, for all large n,

$$H(F(s_{n+1})) - H(F(s_n)) \le [n(\log n)^{1+\varepsilon}t_+(2n^{-\varepsilon}s_n)]^{1/\kappa} \le c_{28}[n^{1-\varepsilon}(\log n)^{1+\varepsilon}s_n]^{1/\kappa}.$$

Recall from Lemma 3.1 and Theorem 1.9 that, almost surely, for all large n, $H[F(s_n)] \ge H[(1-\varepsilon)s_n] \ge \frac{c_{29}s_n^{1/\kappa}}{(\log\log s_n)^{1/\kappa-1}}$, which yields

$$\frac{H(F(s_{n+1}))}{H(F(s_n))} \le 1 + \frac{c_{28}[n^{1-\varepsilon}(\log n)^{1+\varepsilon}s_n]^{1/\kappa}}{c_{29}s_n^{1/\kappa}/(\log\log s_n)^{1/\kappa-1}} \le c_{30}(\log s_n)^{1/\kappa}(\log\log s_n)^{(2+\varepsilon)/\kappa-1}.$$

In view of (6.15), this yields that, almost surely, for large n and $t \in [H(F(s_n)),\ H(F(s_{n+1}))]$, we have

$$\frac{t}{L_X^*(t)} \le \frac{H[F(s_n)]}{L_X^*[H(F(s_n))]}\frac{H(F(s_{n+1}))}{H(F(s_n))} < c_{31}(\log s_n)^{1/\kappa}(\log\log s_n)^{(2+\varepsilon)/\kappa}.$$

Since, almost surely for all large n, $\log H(F(s_n)) \ge \log H((1-\varepsilon)s_n) \ge \frac{1-\varepsilon}{\kappa}\log s_n$ (this is seen first by Lemma 3.1, and then by Theorem 1.9), we have proved that

$$\liminf_{t\to+\infty} \frac{L_X^*(t)}{t(\log t)^{-1/\kappa}(\log\log t)^{-(2+\varepsilon)/\kappa}} \ge c_{32} \qquad \mathbb{P}\text{–a.s.}$$

Since $\varepsilon \in (0, \frac{1}{2})$ is arbitrary, this proves Theorem 1.5 in the case $0 < \kappa < 1$. $\square$

6.2.2 Proof of Theorem 1.4 (case $0 < \kappa < 1$)

By (6.11), for any $s > 0$ and $u > 0$,

$$\begin{aligned}\mathbb{P}(N_\beta > s) &\geq \frac{2}{u}\mathbb{P}\left(\int_0^1 (1-v)^{1/\kappa-2}R^2(v)\mathrm{d}v > s,\ R^2(1)\leq u\right)\\ &\geq \frac{2}{u}\mathbb{P}\left(\int_0^1 (1-v)^{1/\kappa-2}R^2(v)\mathrm{d}v > s\right) - \frac{2}{u}\mathbb{P}\left(R^2(1) > u\right).\end{aligned}$$

The first probability term on the right hand side is taken care of by (6.13), whereas for the second, we have $\frac{1}{u}\log\mathbb{P}(R^2(1) > u) \to -\frac{1}{2}$, for $u\to\infty$. Taking $u := \exp(\sqrt{\log\log r}\,)$ leads to: for any $y > 0$,

$$\liminf_{r\to\infty}\frac{\log\mathbb{P}(N_\beta > y\log\log r)}{\log\log r} \geq -yc_1(\kappa).$$

Plugging this into (6.4) yields that, for $r_n := \exp(n^{1+\varepsilon})$,

$$\sum_{n\geq 1}\mathbb{P}\left(\frac{(H\circ F)(r_{2n})}{(L_X^*\circ H\circ F)(r_{2n})} > \frac{(1-3\varepsilon)\log\log r_{2n}}{\kappa^2 c_1(\kappa)(1+\varepsilon)^3} - \varepsilon\right) = +\infty.$$

Let $R_n := \sum_{k=1}^n r_k$. By Lemma 3.2 (in its notation), almost surely, for infinitely many n,

$$\sup_{u\in[(1-\varepsilon)r_{2n},(1+\varepsilon)r_{2n}]}\frac{H_{X\circ\Theta_{H(R_{2n-1})}}(u)}{(L^*\circ H)_{X\circ\Theta_{H(R_{2n-1})}}(u)} > \frac{(1-8\varepsilon)\log\log r_{2n}}{\kappa^2 c_1(\kappa)}. \tag{6.19}$$

Observe that

$$(L^*\circ H)_{X\circ\Theta_{H(R_{2n-1})}}(u) = \sup_{x\in\mathbb{R}} L_{\widetilde{X}_n}(\widetilde{H}_n(u), x) =: L^*_{\widetilde{X}_n}(\widetilde{H}_n(u)) \tag{6.20}$$

where $(\widetilde{X}_n(v),\ v\geq 0)$ is a diffusion process in the random potential $W_\kappa(x+R_{2n-1}) - W_\kappa(R_{2n-1})$, $x\in\mathbb{R}$, $(L_{\widetilde{X}_n}(t,x),\ t\geq 0, x\in\mathbb{R})$ is its local time and $\widetilde{H}_n(r) := \inf\{t > 0,\ \widetilde{X}_n(t) > r\}$, $r > 0$. Hence, for any $u > 0$, under $\mathbb{P}$, the left hand side of (6.20) is distributed as $L_X^*(H(u))$. Applying (4.5) and Lemma 3.1 to $\widetilde{r}_{2n} := (1-\varepsilon)^2 r_{2n}$, there exists $c_{33} > 0$ such that

$$\sum_n \mathbb{P}\left[L^*_{\widetilde{X}_n}\left(\widetilde{H}_n[(1+\frac{5}{\kappa}(\widetilde{r}_{2n})^{-\delta_0})\widetilde{r}_{2n}]\right) < c_{33}[r_{2n}/\log\log r_{2n}]^{1/\kappa}\right] < \infty.$$

Since $(1+\frac{5}{\kappa}(\widetilde{r}_{2n})^{-\delta_0})\widetilde{r}_{2n} \leq (1-\varepsilon)r_{2n}$ for large n, the Borel–Cantelli lemma gives that, almost surely, for all large n,

$$c_{33}[r_{2n}/\log\log r_{2n}]^{1/\kappa} \leq L^*_{\widetilde{X}_n}\left(\widetilde{H}_n[(1-\varepsilon)r_{2n}]\right) \leq L^*_{\widetilde{X}_n}\left(\widetilde{H}_n(u)\right) \tag{6.21}$$

for any $u \in [(1-\varepsilon)r_{2n}, (1+\varepsilon)r_{2n}]$. Applying Theorem 1.3, we obtain, almost surely, for large n,

$$\begin{aligned} L^*_X[H(R_{2n-1})] &\leq [R_{2n-1}\log^2 R_{2n-1}]^{1/\kappa} \\ &\leq \varepsilon[r_{2n}/\log\log r_{2n}]^{1/\kappa} \\ &\leq (\varepsilon/c_{33})L^*_{\widetilde{X}_n}\left(\widetilde{H}_n(u)\right) \end{aligned}$$

for $u \in [(1-\varepsilon)r_{2n}, (1+\varepsilon)r_{2n}]$, since $R_k \leq k\exp(-k^\varepsilon)r_{k+1}$ for large k. Hence,

$$L^*_X[H(R_{2n-1}+u)] \leq (1+\varepsilon/c_{33})L^*_{\widetilde{X}_n}\left(\widetilde{H}_n(u)\right). \tag{6.22}$$

On the other hand, by Theorem 1.8 and Lemma 3.1, we have, almost surely, for all large n,

$$\log\log r_{2n} \geq (1-\varepsilon)\log\log H(R_{2n-1}+u), \qquad u \in [(1-\varepsilon)r_{2n}, (1+\varepsilon)r_{2n}].$$

Consequently, almost surely for infinitely many n, by (6.22) and (6.19),

$$\begin{aligned} & \inf_{v\in[R_{2n-1}+(1-\varepsilon)r_{2n}, R_{2n-1}+(1+\varepsilon)r_{2n}]} \frac{L^*_X[H(v)]}{H(v)/\log\log H(v)} \\ \leq\ & (1+c_{35}\varepsilon)\inf_{u\in[(1-\varepsilon)r_{2n},(1+\varepsilon)r_{2n}]} \frac{(L^*\circ H)_{X\circ\Theta_{H(R_{2n-1})}}(u)}{H_{X\circ\Theta_{H(R_{2n-1})}}(u)/\log\log r_{2n}} \\ \leq\ & (1+c_{36}\varepsilon)\kappa^2 c_1(\kappa), \end{aligned}$$

proving Theorem 1.4 in the case $0<\kappa<1$. □

6.2.3 Proof of Theorem 1.7

Assume $0<\kappa<1$. Fix $x>0$, and let $r_n := \exp(n^{1+\varepsilon})$. Since $\mathbb{P}(N_\beta < x) > 0$, (6.4) implies

$$\sum_{n\in\mathbb{N}} \mathbb{P}\left(\frac{(H\circ F)(r_{2n})}{(L^*_X\circ H\circ F)(r_{2n})} < \frac{(1+3\varepsilon)x}{\kappa^2} + \varepsilon\right) = +\infty.$$

By Lemma 3.2, almost surely, for infinitely many n,

$$\inf_{u\in[(1-\varepsilon)r_{2n},\ (1+\varepsilon)r_{2n}]} \frac{H_{X\circ\Theta_{H(R_{2n-1})}}(u)}{(L^*\circ H)_{X\circ\Theta_{H(R_{2n-1})}}(u)} < \frac{(1+3\varepsilon)x}{\kappa^2} + \varepsilon. \tag{6.23}$$

With the same notation as in (6.20), $H_{X\circ\Theta_{H(R_{2n-1})}}(u) = H(R_{2n-1}+u) - H(R_{2n-1})$ is the hitting time $\widetilde{H}_n(u)$ of u by the diffusion $\widetilde{X}_n$. For any u, under $\mathbb{P}$, it has the same distribution as $H(u)$. Hence, applying (5.8) and Lemma 3.1 to $\widetilde{r}_{2n} = (1-\varepsilon)^2 r_{2n}$ leads to (for $0<\delta_0<1/2$)

$$\sum_n \mathbb{P}\left[\widetilde{H}_n\left((1+\frac{5}{\kappa}(\widetilde{r}_{2n})^{-\delta_0})\widetilde{r}_{2n}\right) < r_{2n}^{1/\kappa}/\log r_{2n}\right] < \infty.$$

Since $(1+\frac{5}{\kappa}(\widetilde{r}_{2n})^{-\delta_0})\widetilde{r}_{2n} < (1-\varepsilon)r_{2n}$ for large n, it follows from the Borel–Cantelli lemma that, almost surely, for all large n,

$$\frac{r_{2n}^{1/\kappa}}{\log r_{2n}} \le \widetilde{H}_n[(1-\varepsilon)r_{2n}] \le \inf_{u\in[(1-\varepsilon)r_{2n},(1+\varepsilon)r_{2n}]} H_{X\circ\Theta_{H(R_{2n-1})}}(u).$$

On the other hand, by Theorem 1.8, almost surely, for all large n,

$$H(R_{2n-1}) \le [R_{2n-1}\log^2 R_{2n-1}]^{1/\kappa} \le \varepsilon\frac{r_{2n}^{1/\kappa}}{\log r_{2n}}.$$

Hence, for $u\in[(1-\varepsilon)r_{2n},\ (1+\varepsilon)r_{2n}]$, $H(R_{2n-1}+u) \le (1+\varepsilon)H_{X\circ\Theta_{H(R_{2n-1})}}(u)$. Plugging this into (6.23) yields that, almost surely, for infinitely many n,

$$\inf_{u\in[(1-\varepsilon)r_{2n},\ (1+\varepsilon)r_{2n}]} \frac{H(R_{2n-1}+u)}{L_X^*(H(R_{2n-1}+u))} < \frac{(1+\varepsilon)(1+3\varepsilon)x}{\kappa^2} + \varepsilon(1+\varepsilon).$$

This implies $\limsup_{t\to+\infty}\frac{L_X^*(t)}{t} \ge \frac{\kappa^2}{x}$, a.s. Sending $x\to 0$ completes the proof of Theorem 1.7. □

6.3 Case $\kappa = 1$

This section is devoted to the proof of Theorems 1.4 and 1.5 in the case $\kappa=1$ (thus $\lambda=8$).

Let

$$N_\beta(t) := \frac{1}{\sup_{0\le u\le\tau_\beta(8)}\beta(u)}\left[\int_0^1 \frac{L_\beta(\tau_\beta(8),x)-8}{x}\mathrm{d}x + \int_1^{+\infty}\frac{L_\beta(\tau_\beta(8),x)}{x}\mathrm{d}x + 8\log t\right].$$

Exactly as in (6.4), we have, for some $\alpha_1>0$, any $\varepsilon\in(0,\frac{1}{3})$, and all large r,

$$\mathbb{P}\left((1-3\varepsilon)N_{\beta_{t_-(r)}}[t_-(r)] \le \frac{H(F(r))}{L_X^*[H(F(r))]} \le (1+3\varepsilon)N_{\beta_{t_+(r)}}[t_+(r)]\right) \ge 1-\frac{1}{r^{\alpha_1}}, \qquad (6.24)$$

where $t_\pm(\cdot)$ are defined in (3.6). (Compared to (6.4), we no longer have the extra "$\pm\varepsilon$" terms, since they are already taken care of by the presence of $8\log t$ in the definition of $N_\beta(t)$).

With the same notation as in (6.5) and (6.6), the second Ray–Knight theorem (Fact 2.2) gives

$$\begin{aligned} N_\beta(t) &= \frac{1}{\zeta_U}\left[\int_0^1\frac{U^2(x)-8}{x}\mathrm{d}x + \int_1^{+\infty}\frac{U^2(x)}{x}\mathrm{d}x + 8\log t\right] \\ &= \frac{1}{\zeta_U}\left[\int_0^{\zeta_U}\frac{U^2(x)-8}{x}\mathrm{d}x + 8\log\zeta_U + 8\log t\right]. \end{aligned}$$

6.3.1 Proof of Theorem 1.5 (case $\kappa = 1$)

We have $\lambda = 8$ in the case $\kappa = 1$.

Since $\sup_{x>0} \frac{\log x}{x} < \infty$, we have

$$N_\beta(t) \le c_{37} + \frac{1}{\zeta_U} \int_0^{\zeta_U} \frac{|U^2(x) - 8|}{x} \mathrm{d}x + \frac{8 \log t}{\zeta_U}.$$

We claim that for some constant $c_{38} > 0$,

$$\limsup_{y \to +\infty} \frac{1}{y} \log \mathbb{P} \left(\frac{1}{\zeta_U} \int_0^{\zeta_U} \frac{|U^2(x) - 8|}{x} \mathrm{d}x > y \right) \le -c_{38}. \tag{6.25}$$

Indeed, $\zeta_U = \sup_{0 \le u \le \tau_\beta(8)} \beta(u)$ by definition (see (6.6)), which, in view of (4.1), implies that $\mathbb{P}(\zeta_U > z) = 1 - e^{-4/z} \le \frac{4}{z}$ for $z > 0$. Therefore, if we write $p(y)$ for the probability expression at (6.25), we have, for any $z > 0$,

$$p(y) \le \frac{4}{z} + \mathbb{P} \left(\frac{1}{\zeta_U} \int_0^{\zeta_U} \frac{|U^2(x) - 8|}{x} \mathrm{d}x > y, \ \zeta_U \le z \right).$$

In the notation of (6.8)–(6.10), this yields

$$\begin{aligned} p(y) &\le \frac{4}{z} + \mathbb{P} \left(\frac{1}{\gamma_a} \int_0^1 \frac{|R^2(\gamma_a v) - 8|}{1 - v} \mathrm{d}v > y, \ \gamma_a \le z \right) \\ &= \frac{4}{z} + \mathbb{E} \left(\frac{2}{R^2(1)} \mathbf{1}_{\{\int_0^1 \frac{|R^2(v) - R^2(1)|}{1-v} \mathrm{d}v > y, \ R^2(1) \ge 8/z\}} \right) \\ &\le \frac{4}{z} + \frac{z}{4} \mathbb{P} \left(\int_0^1 \frac{|R^2(v) - R^2(1)|}{1 - v} \mathrm{d}v > y \right). \end{aligned} \tag{6.26}$$

In order to apply Schilder's theorem as in (6.13), let $\phi \in \mathcal{C}_0$. By the Cauchy–Schwarz inequality, $|\phi(t)| \le \sqrt{t}\, [\int_0^1 \phi'(u)^2 \mathrm{d}u]^{1/2}$, and $|\phi(1-t) - \phi(1)|$ satisfies a similar inequality. Hence,

$$\begin{aligned} \int_0^1 \frac{|\phi^2(u) - \phi^2(1)|}{1 - u} \mathrm{d}u &\le \int_0^1 \frac{|\phi(u) - \phi(1)|}{1 - u} [|\phi(u)| + |\phi(1)|] \mathrm{d}u \\ &\le 2 \left(\int_0^1 \frac{\mathrm{d}u}{\sqrt{1 - u}} \right) \int_0^1 \phi'(u)^2 \mathrm{d}u. \end{aligned}$$

Consequently,

$$c_{39} := \inf \left\{ \frac{1}{2} \int_0^1 \phi'(u)^2 \mathrm{d}u : \quad \phi \in \mathcal{C}_0, \ \int_0^1 \frac{|\phi^2(u) - \phi^2(1)|}{1 - u} \mathrm{d}u > 1 \right\} > 0.$$

Applying Schilder's theorem gives that

$$\limsup_{y\to+\infty} \frac{1}{y} \log \mathbb{P}\left(\int_0^1 \frac{|R^2(v)-R^2(1)|}{1-v} \mathrm{d}v > y\right) \le -c_{39}.$$

Plugging this into (6.26), and taking $z = \exp(\frac{c_{39}}{2}y)$ there, we obtain the claimed inequality in (6.25), with $c_{38} := \frac{c_{39}}{2}$.

On the other hand, by (4.1),

$$\mathbb{P}\left(\frac{8\log t}{\zeta_U} > 2(1+2\varepsilon)(\log t)\log\log t\right) = \frac{1}{(\log t)^{1+2\varepsilon}}.$$

Therefore, for all large t,

$$\mathbb{P}\left\{N_\beta(t) > 2(1+3\varepsilon)(\log t)\log\log t\right\} \le \frac{2}{(\log t)^{1+2\varepsilon}}.$$

Let $s_n := \exp(n^{1-\varepsilon})$. By (6.24),

$$\sum_{n=1}^{+\infty} \mathbb{P}\left(\frac{H(F(s_n))}{L_X^*[H(F(s_n))]} > 2(1+3\varepsilon)^2(\log s_n)\log\log s_n\right) < \infty,$$

which, by means of the Borel–Cantelli lemma, implies that, almost surely, for all large n,

$$\frac{H(F(s_n))}{L_X^*[H(F(s_n))]} \le 2(1+3\varepsilon)^2(\log s_n)\log\log s_n. \tag{6.27}$$

Now we give an upper bound for $\frac{H(F(s_{n+1}))}{H(F(s_n))}$. By Lemma 3.1, almost surely for n large enough, $F(s_n) \ge (1-\varepsilon)s_n$. An application of Theorem 1.9 yields that, almost surely, for large n,

$$H(F(s_n)) \ge H[(1-\varepsilon)s_n] \ge 4(1-2\varepsilon)s_n \log s_n. \tag{6.28}$$

With the same notation and the same arguments as in (6.16) and (6.17), almost surely for all large n, $H(F(s_{n+1})) - H(F(s_n)) \le \widehat{H}_n[(2-\varepsilon)n^{-\varepsilon}s_n]$. Moreover, $\widehat{H}_n(r)$ is distributed as $H(r)$ under $\mathbb{P}$ for any $r > 0$. Hence, applying Lemma 3.1 and (5.6) to $r = \widetilde{s}_n := 2n^{-\varepsilon}s_n$ and $a(e^{-2}\widetilde{s}_n) = 8n(\log n)^{1+\varepsilon}$ for $0 < \delta_0 < \frac{1}{2}$, we get

$$\sum_n \mathbb{P}\left[\widehat{H}_n\left((1-\frac{5}{\kappa}(\widetilde{s}_n)^{-\delta_0})\widetilde{s}_n\right) > 32(1+\varepsilon)t_+(\widetilde{s}_n)[c_{10} + n(\log n)^{1+\varepsilon} + \log t_+(\widetilde{s}_n)]\right] < \infty.$$

Since $[1 - \frac{5}{\kappa}(\widetilde{s}_n)^{-\delta_0}]\widetilde{s}_n \ge (2-\varepsilon)n^{-\varepsilon}s_n$ (for large n), the Borel–Cantelli lemma yields that, almost surely, for large n,

$$\widehat{H}_n((2-\varepsilon)n^{-\varepsilon}s_n) \le 32(1+\varepsilon)t_+(2n^{-\varepsilon}s_n)[c_{10} + n(\log n)^{1+\varepsilon} + \log t_+(2n^{-\varepsilon}s_n)].$$

Hence,

$$H(F(s_{n+1})) - H(F(s_n)) \leq c_{40} s_n (\log s_n)(\log n)^{1+\varepsilon}.$$

Consequently, almost surely for large n,

$$\frac{H(F(s_{n+1}))}{H(F(s_n))} \leq (c_{41} + \varepsilon)(\log\log s_n)^{1+\varepsilon}.$$

Let $t \in [H(F(s_n)),\, H(F(s_{n+1}))]$. By (6.27),

$$\frac{t}{L_X^*(t)} \leq \frac{H[F(s_n)]}{L_X^*[H(F(s_n))]} \frac{H(F(s_{n+1}))}{H(F(s_n))} < 3c_{41} (\log s_n)(\log\log s_n)^{2+\varepsilon}.$$

Since, almost surely for large n, $\log H(F(s_n)) \geq \log H((1-\varepsilon)s_n) \geq \log s_n$ (by Lemma 3.1 and Theorem 1.9), this yields

$$\liminf_{t\to+\infty} \frac{L_X^*(t)}{t/[(\log t)(\log\log t)^{2+\varepsilon}]} \geq \frac{1}{3c_{41}} \qquad \mathbb{P}\text{–a.s.}$$

Theorem 1.5 is proved in the case $\kappa = 1$. □

6.3.2 Proof of Theorem 1.4 (case $\kappa = 1$)

Again, $\lambda = 8$. Recall that

$$N_\beta(t) = \frac{1}{\zeta_U}\left[\int_0^{\zeta_U} \frac{U^2(x) - 8}{x} \mathrm{d}x + 8\log\zeta_U + 8\log t\right].$$

This time, we need to bound $N_\beta(t)$ from below. Recall that $\zeta_U = \sup_{0\leq u\leq \tau_\beta(8)} \beta(u)$. By (4.1), for $z > 8e$,

$$\mathbb{P}\left(8\frac{\log\zeta_U}{\zeta_U} < -z\right) \leq \mathbb{P}\left(\zeta_U < \frac{\log(z/8)}{z/8}\right) = \exp\left(-\frac{z}{2\log(z/8)}\right).$$

By (4.1) again,

$$\mathbb{P}\left(\frac{8\log t}{\zeta_U} > 2(1-\varepsilon)(\log t)\log\log t\right) = \frac{1}{(\log t)^{1-\varepsilon}}.$$

On the other hand, for all large y, $\mathbb{P}(\frac{1}{\zeta_U}\int_0^{\zeta_U} \frac{|U^2(x)-8|}{x}\mathrm{d}x > y) \leq e^{-c_{42}y}$ (see (6.25)). Assembling these pieces yields that, for all large t,

$$\mathbb{P}[N_\beta(t) > 2(1-2\varepsilon)(\log t)\log\log t] \geq \frac{1}{2(\log t)^{1-\varepsilon}}.$$

Let $r_n := \exp(n^{1+\varepsilon})$. In view of (6.24), we get

$$\sum_{n\in\mathbb{N}} \mathbb{P}\left(\frac{(H\circ F)(r_{2n})}{(L^*\circ H\circ F)(r_{2n})} > 2(1-2\varepsilon)(1-3\varepsilon)(\log r_{2n})\log\log r_{2n}\right) = +\infty.$$

It follows from Lemma 3.2 that, almost surely, for infinitely many n,

$$\sup_{u\in[(1-\varepsilon)r_{2n},\,(1+\varepsilon)r_{2n}]} \frac{H_{X\circ\Theta_{H(R_{2n-1})}}(u)}{(L^*\circ H)_{X\circ\Theta_{H(R_{2n-1})}}(u)} > 2(1-2\varepsilon)(1-3\varepsilon)(\log r_{2n})\log\log r_{2n}. \tag{6.29}$$

The expression on the left hand side of (6.29) is "close to" $H(r_{2n})/L_X^*[H(r_{2n})]$, but we need to prove this rigorously. With the same argument as in the displays between (6.20) and (6.21), we get that, almost surely, for large n,

$$\inf_{u\in[(1-\varepsilon)r_{2n},(1+\varepsilon)r_{2n}]}(L^*\circ H)_{X\circ\Theta_{H(R_{2n-1})}}(u) \ge c_{43}r_{2n}/\log\log r_{2n}.$$

Observe that $R_k \le k\exp(-k^\varepsilon)r_{k+1}$ (for large k). Exactly as in the case $0<\kappa<1$, we apply Theorem 1.3, to see that, almost surely, for large n,

$$\begin{aligned} L_X^*[H(R_{2n-1})] &\le R_{2n-1}\log^2 R_{2n-1} \\ &\le \varepsilon r_{2n}/\log\log r_{2n} \\ &\le (\varepsilon/c_{43})\inf_{u\in[(1-\varepsilon)r_{2n},(1+\varepsilon)r_{2n}]}(L^*\circ H)_{X\circ\Theta_{H(R_{2n-1})}}(u), \end{aligned}$$

which implies, for all $u\in[(1-\varepsilon)r_{2n},(1+\varepsilon)r_{2n}]$,

$$L_X^*[H(R_{2n-1}+u)] \le (1+\varepsilon/c_{43})(L^*\circ H)_{X\circ\Theta_{H(R_{2n-1})}}(u). \tag{6.30}$$

By Theorem 1.8, almost surely for all large n, $\sup_{u\in[(1-\varepsilon)r_{2n},(1+\varepsilon)r_{2n}]}\log H(R_{2n-1}+u) \le (1+\varepsilon)\log r_{2n}$. In view of (6.29), there are almost surely infinitely many n such that

$$\begin{aligned} &\inf_{v\in[R_{2n-1}+(1-\varepsilon)r_{2n},R_{2n-1}+(1+\varepsilon)r_{2n}]}\frac{L_X^*[H(v)]}{H(v)/[(\log H(v))\log\log H(v)]} \\ \le\ &(1+c_{44}\varepsilon)\inf_{u\in[(1-\varepsilon)r_{2n},(1+\varepsilon)r_{2n}]}\frac{(L_X^*\circ H)_{X\circ\Theta_{H(R_{2n-1})}}(u)}{H_{X\circ\Theta_{H(R_{2n-1})}}(u)[(\log r_{2n})\log\log r_{2n}]^{-1}} \\ \le\ &(1+c_{45}\varepsilon)/2. \end{aligned}$$

This proves Theorem 1.4 in the case $\kappa=1$. □

7 Proof of Lemma 3.3

This section is devoted to the proof of Lemma 3.3. The basic idea goes back to Hu et al. [13], but requires considerable refinements due to the complicated nature of the process $x\mapsto L_X(t,x)$.

Let $\kappa>0$. Recall $A(x)=\int_0^x e^{W_\kappa(u)}\mathrm{d}u$, and $A_\infty=\lim_{x\to+\infty}A(x)<\infty$, a.s. As in Brox [6], the general diffusion theory leads to

$$X(t)=A^{-1}[B(T^{-1}(t))], \qquad t\ge 0,$$

where B is a Brownian motion independent of W, and

$$T(r) := \int_0^r \exp\{-2W_\kappa[A^{-1}(B(s))]\}\mathrm{d}s, \qquad 0 \le r < \sigma_B(A_\infty),$$

(A^{-1} and T^{-1} denote respectively the inverses of A and T). The local time of X can be written as

$$L_X(t,x) = e^{-W_\kappa(x)} L_B(T^{-1}(t), A(x)), \qquad t \ge 0,\ x \in \mathbb{R}. \tag{7.1}$$

Let $H(\cdot)$ be the first hitting time of X as in (1.3). Then

$$\begin{aligned} H(r) &= T[\sigma_B(A(r))] \\ &= \int_{-\infty}^{A(r)} e^{-2W_\kappa[A^{-1}(x)]} L_B(\sigma_B[A(r)], x)\mathrm{d}x \\ &= H_-(r) + H_+(r), \end{aligned}$$

where

$$\begin{aligned} H_-(r) &:= \int_{-\infty}^{0} \exp\{-2W_\kappa[A^{-1}(x)]\} L_B\{\sigma_B[A(r)], x\}\mathrm{d}x, \\ H_+(r) &:= \int_{0}^{A(r)} \exp\{-2W_\kappa[A^{-1}(x)]\} L_B\{\sigma_B[A(r)], x\}\mathrm{d}x. \end{aligned}$$

In view of the first Ray–Knight theorem (Fact 2.1), it is more convenient to study $L_X^*(H(r))$ instead of $L_X^*(t)$. We consider

$$L_X^+(H(r)) := \sup_{x\ge 0} L_X(H(r), x) = \sup_{x \ge 0} \left\{ e^{-W_\kappa(x)} L_B[\sigma_B(A(r)), A(x)] \right\}.$$

Recall F from (3.1) and notice that $F(r) > 0$ on $E_1(r)$. By the first Ray–Knight theorem (Fact 2.1), there exists a squared Bessel process of dimension 2, starting from 0 and denoted by $(R_2^2(t),\ t \ge 0)$, independent of W_κ, such that

$$\left(\frac{L_B\{\sigma_B[A(F(r))], A(F(r)) - sA(F(r))\}}{A(F(r))},\ s \in [0,1] \right) = \left(R_2^2(s),\ s \in [0,1]\right).$$

Therefore,

$$L_X^+[H(F(r))] = \sup_{x \in [0, F(r)]} \left\{ e^{-W_\kappa(x)} A(F(r)) R_2^2 \left[\frac{A(F(r)) - A(x)}{A(F(r))} \right] \right\}.$$

Moreover, by Lamperti's representation theorem (Fact 2.3), there exists a Bessel process $\rho = (\rho(t),\ t \ge 0)$, of dimension $(2 - 2\kappa)$, starting from $\rho(0) = 2$, such that

$$\forall t \ge 0, \qquad e^{W_\kappa(t)/2} = \frac{1}{2}\rho(A(t)).$$

Now, let

$$\widetilde{R}_{2+2\kappa}(t) := \rho(A_\infty - t), \qquad 0 \le t \le A_\infty.$$

By Williams' time reversal theorem (Fact 2.4), $\widetilde{R}_{2+2\kappa}$ is a Bessel process of dimension $(2+2\kappa)$, starting from 0. Since $A(F(r))$ s independent of R_2, $u \mapsto \sqrt{A(F(r))}R_2(u/A(F(r)))$ is a 2–dimensional Bessel process, starting from 0 and independent of $\widetilde{R}_{2+2\kappa}$. We still denote by R_2 this new Bessel process. We obtain

$$\begin{aligned} L_X^+[H(F(r))] &= 4 \sup_{x\in[0,F(r)]} \frac{R_2^2[A(F(r)) - A(x)]}{\widetilde{R}^2_{2+2\kappa}[A_\infty - A(x)]} \\ &= 4 \sup_{v\in[0,A(F(r))]} \frac{R_2^2(v)}{\widetilde{R}^2_{2+2\kappa}[A_\infty - A(F(r)) + v]}. \end{aligned}$$

Doing the same transformations on $H_+(r)$ and recalling that $A_\infty - A(F(r)) = \delta(r) = \exp(-\kappa r/2)$, we obtain

$$\begin{aligned} & (L_X^+[H(F(r))], H_+[F(r)]) \\ = & \left(4 \sup_{v\in[0,A(F(r))]} \frac{R_2^2(v)}{\widetilde{R}^2_{2+2\kappa}[\delta(r)+v]}, 16\int_0^{A(F(r))} \frac{R_2^2(s)}{\widetilde{R}^4_{2+2\kappa}[\delta(r)+s]}\mathrm{d}s\right) \\ = & \left(4 \sup_{v\in[0,\delta(r)^{-1}A(F(r))]} \frac{R_2^2[\delta(r)v]}{\widetilde{R}^2_{2+2\kappa}[\delta(r)(1+v)]}, 16\int_0^{\delta(r)^{-1}A(F(r))} \frac{R_2^2[\delta(r)u]\delta(r)\mathrm{d}u}{\widetilde{R}^4_{2+2\kappa}[\delta(r)(1+u)]}\right). \end{aligned}$$

We still denote by R_2 the 2-dimensional Bessel process $u \mapsto \frac{1}{\sqrt{\delta(r)}}R_2(\delta(r)u)$. We define

$$\widehat{R}_{2+2\kappa}(u) = \frac{1}{\sqrt{\delta(r)}}\widetilde{R}_{2+2\kappa}[\delta(r)(1+u)], \qquad u \ge 0.$$

Notice that $(\widehat{R}_{2+2\kappa}(u),\ u \ge 0)$ is a $(2+2\kappa)$–dimensional Bessel process, starting from $\widetilde{R}_{2+2\kappa}(\delta(r))/\sqrt{\delta(r)}$ and independent of R_2.

Recall (Karlin and Taylor [14] p. 335) that a Jacobi process of dimension (d_1, d_2) is a solution of the stochastic differential equation

$$\mathrm{d}Y(t) = 2\sqrt{Y(t)(1-Y(t))}\mathrm{d}\widehat{\beta}(t) + [d_1 - (d_1+d_2)Y(t)]\mathrm{d}t, \tag{7.2}$$

where $\widehat{\beta}$ is a standard Brownian motion.

According to Warren and Yor [28], there exists a Jacobi process $(Y(t),\ t \ge 0)$ of dimension $(2, 2+2\kappa)$, starting from 0, independent of $(R_2^2(t) + \widehat{R}^2_{2+2\kappa}(t),\ t \ge 0)$, such that

$$\forall u \ge 0, \qquad \frac{R_2^2(u)}{R_2^2(u) + \widehat{R}^2_{2+2\kappa}(u)} = Y \circ \Lambda_Y(u),$$

where

$$\Lambda_Y(u) := \int_0^u \frac{\mathrm{d}s}{R_2^2(s) + \widehat{R}_{2+2\kappa}^2(s)}. \tag{7.3}$$

In particular, $(\Lambda_Y(t),\ t \geq 0)$ is independent of Y. As a consequence, for all $r \geq 0$,

$$\begin{aligned}
&(L_X^+[H(F(r))], H_+[F(r)]) \\
&= \left(4 \sup_{u\in[0,\delta(r)^{-1}A(F(r))]} \frac{Y\circ\Lambda_Y(u)}{1-Y\circ\Lambda_Y(u)}, 16\int_0^{\delta(r)^{-1}A(F(r))} \frac{[Y\circ\Lambda_Y(u)]\Lambda_Y'(u)\mathrm{d}u}{[1-Y\circ\Lambda_Y(u)]^2}\right) \\
&= \left(4 \sup_{u\in[0,\gamma(r)]} \frac{Y(u)}{1-Y(u)}, 16\int_0^{\gamma(r)} \frac{Y(u)}{(1-Y(u))^2}\mathrm{d}u\right),
\end{aligned}$$

where

$$\gamma(r) := \Lambda_Y[\delta(r)^{-1}A(F(r))]. \tag{7.4}$$

Let $\alpha_\kappa := 1/(4+2\kappa)$ and let $T_Y(\alpha_\kappa) := \inf\{t > 0,\ Y(t) = \alpha_\kappa\}$ be the hitting time of α_κ by Y. Define

$$\begin{aligned}
\overline{L}(r) &:= 4 \sup_{u\in[0,T_Y(\alpha_\kappa)]} \frac{Y(u)}{1-Y(u)}, & \overline{H}(r) &:= 16 \int_0^{T_Y(\alpha_\kappa)} \frac{Y(u)}{(1-Y(u))^2}\mathrm{d}u, \\
L_0(r) &:= 4 \sup_{u\in[T_Y(\alpha_\kappa),\gamma(r)]} \frac{Y(u)}{1-Y(u)}, & H_0(r) &:= 16 \int_{T_Y(\alpha_\kappa)}^{\gamma(r)} \frac{Y(u)}{(1-Y(u))^2}\mathrm{d}u.
\end{aligned} \tag{7.5}$$

We have

$$\left(L_X^+[H(F(r))],\ H_+(F(r))\right) = \left(\max\{\overline{L}(r), L_0(r)\},\ \overline{H}(r) + H_0(r)\right). \tag{7.6}$$

Moreover, on the event $\{T_Y(\alpha_\kappa) \leq 64 \log r\}$, we have

$$\overline{L}(r) \leq \frac{4\alpha_\kappa}{1-\alpha_\kappa} \qquad \text{and} \qquad \overline{H}(r) \leq \frac{2^{10}\alpha_\kappa}{(1-\alpha_\kappa)^2} \log r. \tag{7.7}$$

Observe that a scale function of Y is $S(y) := \int_{\alpha_\kappa}^y \frac{\mathrm{d}x}{x(1-x)^{1+\kappa}}$. There exists a Brownian motion $(\beta(t),\ t \geq 0)$ such that

$$Y[t + T_Y(\alpha_\kappa)] = S^{-1}\{\beta[U(t)]\}, \qquad t \geq 0,$$

where

$$U(t) := 4\int_0^t \frac{\mathrm{d}s}{Y[s+T_Y(\alpha_\kappa)]\{1-Y[s+T_Y(\alpha_\kappa)]\}^{1+2\kappa}}.$$

The rest of the proof of Lemma 3.3 requires some more estimates, stated as Lemmas 7.1–7.4 below. We admit these lemmas for the moment, and complete the proof of Lemma 3.3.

The proof of Lemmas 7.1–7.4 will be presented, separately, in forthcoming subsections.

Lemma 7.1 *Let $(R(t),\ t \geq 0)$ be a Bessel process of dimension $d > 4$, starting from $R_0 \stackrel{\mathcal{L}}{=} \widetilde{R}_{d-2}(1)$, where $(\widetilde{R}_{d-2}(t),\ t \in [0,1])$ is $(d-2)$–dimensional Bessel process. For any $\delta_3 \in (0, \frac{1}{2})$ and all large t,*

$$\mathbb{P}\left\{\left|\frac{1}{\log t}\int_0^t \frac{\mathrm{d}s}{R^2(s)} - \frac{1}{d-2}\right| > \frac{1}{(\log t)^{(1/2)-\delta_3}}\right\} \leq \exp\left(-c_{46}\,(\log t)^{2\delta_3}\right).$$

Lemma 7.2 *Let $\delta_1 > 0$ and define*

$$E_{10} := \left\{\tau_\beta[(1-v^{-\delta_1})\lambda v] \leq U(v) \leq \tau_\beta[(1+v^{-\delta_1})\lambda v]\right\}. \tag{7.8}$$

If δ_1 is small enough, then for all large v,

$$\mathbb{P}(E_{10}^c) \leq \frac{1}{v^{1/4-5\delta_1}}. \tag{7.9}$$

Lemma 7.3 *Let $\kappa > 0$ and define*

$$\begin{aligned} L_X^{*-}(+\infty) &:= \sup_{r\geq 0}\ \sup_{x<0} L_X(H(r),x) = \sup_{t\geq 0}\ \sup_{x<0} L_X(t,x), \\ H_-(+\infty) &:= \lim_{r\to+\infty} H_-(r). \end{aligned}$$

There exist $c_{47} > 0$ and $c_{48} > 0$ such that for all large z,

$$\mathbb{P}\left(L_X^{*-}(+\infty) > z\right) \leq \frac{c_{47}}{z^{\kappa/(\kappa+2)}}, \tag{7.10}$$

$$\mathbb{P}(H_-(+\infty) > z) \leq \frac{c_{48}}{(z/\log z)^{\kappa/(\kappa+2)}}. \tag{7.11}$$

Lemma 7.4 *Let $(\beta(t),\ t \geq 0)$ be a Brownian motion, and let $\lambda = 4(1+\kappa)$. We define*

$$J_\beta(\kappa,t,\lambda) := \int_0^1 y(1-y)^{\kappa-2} L_\beta(\tau_\beta(\lambda), \frac{S(y)}{t})\,dy, \qquad 0 < \kappa \leq 1,\ t \geq 0. \tag{7.12}$$

Let $0 < d < 1$ and let $0 < \varepsilon < 1$.
(i) *Case $0 < \kappa < 1$: there exist $c_{49} > 0$ and $c_{50} > 0$ such that for all large t, on an event E_{11} of probability greater than $1 - c_{49}/t^d$, we have*

$$(1-\varepsilon)K_\beta(\kappa) - c_{50}t^{1-1/\kappa} \leq \kappa^{2-1/\kappa}t^{1-1/\kappa}J_\beta(\kappa,t,\lambda) \leq (1+\varepsilon)K_\beta(\kappa) + c_{50}t^{1-1/\kappa}, \tag{7.13}$$

where $K_\beta(\kappa)$ is defined in (3.7).
(ii) *Case $\kappa = 1$: there exists $c_{49} > 0$ such that for t large enough, on an event E_{11} of probability greater than $1 - c_{49}/t^d$,*

$$(1-\varepsilon)[C_\beta + 8\log t] \leq J_\beta(1,t,8) \leq (1+\varepsilon)[C_\beta + 8\log t], \tag{7.14}$$

where C_β is defined in (3.8).

By admitting Lemmas 7.1–7.4, we can now complete the proof of Lemma 3.3.

Proof of Lemma 3.3: part (i). Notice that

$$S(y) \underset{y\to 1}{\sim} \int_{\alpha_\kappa}^{y} \frac{\mathrm{d}s}{(1-s)^{1+\kappa}} \underset{y\to 1}{\sim} \frac{1}{\kappa}\frac{1}{(1-y)^\kappa}. \tag{7.15}$$

This yields

$$\frac{y}{1-y} \underset{y\to 1}{\sim} [\kappa S(y)]^{1/\kappa}. \tag{7.16}$$

Let

$$\begin{aligned}\widetilde{L}_0(r) &:= 4 \sup_{u\in[T_Y(\alpha_\kappa),\gamma(r)]} [\kappa S(Y(u))]^{1/\kappa} \\ &= 4 \sup_{u\in[0,\gamma(r)-T_Y(\alpha_\kappa)]} [\kappa\beta(U(u))]^{1/\kappa} \\ &= 4 \sup_{t\in[0,U(\gamma(r)-T_Y(\alpha_\kappa))]} [\kappa\beta(t)]^{1/\kappa}. \end{aligned} \tag{7.17}$$

Let $\varepsilon > 0$ and recall L_0 from (7.5). By (7.16), there exists a constant $c_{51} > 0$ depending on ε such that

$$\left\{\widetilde{L}_0(r) > c_{51}\right\} \subset \left\{(1-\varepsilon)\widetilde{L}_0(r) \le L_0(r) \le (1+\varepsilon)\widetilde{L}_0(r)\right\}. \tag{7.18}$$

We look for an estimate of $U[\gamma(r)-T_Y(\alpha_\kappa)]$ appearing in the definition of $\widetilde{L}_0(r)$ in (7.17). Recall (Dufresnes [9]) that $A_\infty \overset{\mathcal{L}}{=} 2/\gamma_\kappa$, where γ_κ is a gamma variable of parameter κ. Since $A(F(r)) \le A_\infty$, we have

$$\mathbb{P}[A(F(r)) > r^{2/\kappa}] \le \mathbb{P}[\gamma_\kappa < 2r^{-2/\kappa}] \le \frac{2^\kappa r^{-2}}{\kappa\Gamma(\kappa)}.$$

On the other hand, by definition, $A(F(r)) = A_\infty - \delta(r) = A_\infty - e^{-\kappa r/2}$ (see 3.1), which implies

$$\mathbb{P}\left[A(F(r)) < \frac{1}{2\log r}\right] \le \mathbb{P}\left[\frac{2}{\gamma_\kappa} < \frac{1}{2\log r} + \delta(r)\right] \le \frac{1}{\Gamma(\kappa)r^2}.$$

Consequently, for large r,

$$\mathbb{P}\left\{\frac{\kappa}{2}r - 2\log\log r \le \log[\delta(r)^{-1}A(F(r))] \le \frac{\kappa}{2}r + \frac{2}{\kappa}\log r\right\} \ge 1 - \frac{c_{52}}{r^2}.$$

Recall that $\gamma(r) = \Lambda_Y[\delta(r)^{-1}A(F(r))]$, see (7.4). Thus, for large r,

$$\mathbb{P}\left\{\Lambda_Y[\exp(\frac{\kappa}{2}r - 2\log\log r)] \le \gamma(r) \le \Lambda_Y[\exp(\frac{\kappa}{2}r + \frac{2}{\kappa}\log r)]\right\} \ge 1 - \frac{c_{52}}{r^2}.$$

By definition, $\Lambda_Y(u) = \int_0^u \frac{\mathrm{d}s}{R_2^2(s)+\widehat{R}^2_{2+2\kappa}(s)}$. Since $(R_2^2(t)+\widehat{R}^2_{2+2\kappa}(t),\ t\geq 0)$ is a $(4+2\kappa)$–dimensional squared Bessel process starting from $\widetilde{R}^2_{2+2\kappa}(\delta(r))/\delta(r)$, it follows from Lemma 7.1 that there exist constants $\delta_3 \in (0, \frac12)$, $c_{53}>0$ and $c_{54}>0$, such that for large r,

$$\mathbb{P}\left\{ \frac{\kappa}{\lambda} r - c_{53} r^{1/2+\delta_3} \leq \gamma(r) \leq \frac{\kappa}{\lambda} r + c_{53} r^{1/2+\delta_3} \right\} \geq 1 - \frac{c_{54}}{r^2}, \tag{7.19}$$

where $\lambda = 4(1+\kappa)$, as before.

To study the behaviour of $T_Y(\alpha_\kappa)$, we go back to the stochastic differential equation in (7.2) satisfied by the Jacobi process $Y(\cdot)$, with $d_1 = 2$ and $d_2 = 2+2\kappa$. By the Dubins–Schwarz theorem, there exists a Brownian motion $(\widehat{B}(t),\ t\geq 0)$ such that

$$Y(t) = \widehat{B}\left(4\int_0^t Y(s)(1-Y(s))\mathrm{d}s\right) + \int_0^t [2-(4+2\kappa)Y(s)]\mathrm{d}s, \qquad t\geq 0.$$

Recall that $\alpha_\kappa = 1/(4+2\kappa)$. Let $t \geq 2\alpha_\kappa$. Since $Y(s)\in(0,1)$ for any $s\geq 0$, we have, on the event $\{T_Y(\alpha_\kappa)\geq t\}$,

$$\inf_{0\leq s\leq 4t} \widehat{B}(s) \leq \widehat{B}\left(4\int_0^t Y(s)(1-Y(s))\mathrm{d}s\right) \leq \alpha_\kappa - t \leq -\frac{t}{2}.$$

As a consequence, for $t\geq 2\alpha_\kappa$,

$$\mathbb{P}(T_Y(\alpha_\kappa) > t) \leq \mathbb{P}\left(\inf_{0\leq s\leq 4t} \widehat{B}(s) \leq -\frac{t}{2}\right) \leq 2\exp\left(-\frac{t}{32}\right). \tag{7.20}$$

In particular, $\mathbb{P}[T_Y(\alpha_\kappa) > 64\log r] \leq \frac{2}{r^2}$ for large r. Plugging this into (7.19) yields that for large r,

$$\mathbb{P}\left\{ \frac{\kappa}{\lambda} r - c_{55} r^{1/2+\delta_3} \leq \gamma(r) - T_Y(\alpha_\kappa) \leq \frac{\kappa}{\lambda} r + c_{53} r^{1/2+\delta_3} \right\} \geq 1 - \frac{c_{56}}{r^2}.$$

Let $\underline{\gamma} = \underline{\gamma}(r) := \frac{\kappa}{\lambda} r - c_{55} r^{1/2+\delta_3}$ and $\overline{\gamma} = \overline{\gamma}(r) := \frac{\kappa}{\lambda} r + c_{53} r^{1/2+\delta_3}$. Then, for large r,

$$\mathbb{P}\left\{ U(\underline{\gamma}) \leq U[\gamma(r) - T_Y(\alpha_\kappa)] \leq U(\overline{\gamma}) \right\} \geq 1 - \frac{c_{56}}{r^2}.$$

By Lemma 7.2, for small $\delta_1>0$ and all large r,

$$\mathbb{P}\left\{ \tau_\beta\left[(1-\frac{1}{\underline{\gamma}^{\delta_1}})\lambda\underline{\gamma}\right] \leq U[\gamma(r) - T_Y(\alpha_\kappa)] \leq \tau_\beta\left[(1+\frac{1}{\overline{\gamma}^{\delta_1}})\lambda\overline{\gamma}\right] \right\} \geq 1 - \frac{1}{r^{c_{57}}}.$$

We choose δ_1 so small that $\delta_1 < 1/2 - \delta_3$. Then for large r, we have $(1-\frac{1}{\underline{\gamma}^{\delta_1}})\lambda\underline{\gamma} \geq [1 - 2(\frac{\lambda}{\kappa})^{\delta_1} r^{-\delta_1}]\kappa r$, and $(1+\frac{1}{\overline{\gamma}^{\delta_1}})\lambda\overline{\gamma} \leq [1 + 2(\frac{\lambda}{\kappa})^{\delta_1} r^{-\delta_1}]\kappa r$. Recall that $\lambda t_\pm(r) = [1 \pm 2(\frac{\lambda}{\kappa})^{\delta_1} r^{-\delta_1}]\kappa r$ (see (3.6)). Therefore, for large r,

$$\mathbb{P}\left\{ \tau_\beta[\lambda t_-(r)] \leq U[\gamma(r) - T_Y(\alpha_\kappa)] \leq \tau_\beta[\lambda t_+(r)] \right\} \geq 1 - \frac{1}{r^{c_{57}}}. \tag{7.21}$$

Plugging this into (7.17) gives, for large r,

$$\mathbb{P}\left\{4 \sup_{t\in[0,\tau_\beta(\lambda t_-(r))]}[\kappa\beta(t)]^{1/\kappa} \le \widetilde{L}_0(r) \le 4 \sup_{t\in[0,\tau_\beta(\lambda t_+(r))]}[\kappa\beta(t)]^{1/\kappa}\right\} \ge 1-\frac{1}{r^{c_{57}}}.$$

Since $\widehat{L}_\pm(r) = 4\sup_{t\in[0,\tau_\beta(\lambda t_\pm(r))]}[\kappa\beta(t)]^{1/\kappa}$ by definition (see (3.10)), we have, for large r,

$$\mathbb{P}\left\{\widehat{L}_-(r) \le \widetilde{L}_0(r) \le \widehat{L}_+(r)\right\} \ge 1-\frac{1}{r^{c_{57}}}.$$

By (4.2), $\mathbb{P}\{\widehat{L}_-(r) > r^{(1-\delta_1)/\kappa}\} \ge 1-\frac{1}{r}$, for all large r. Applying (7.18), we obtain: for large r,

$$\mathbb{P}\left\{(1-\varepsilon)r^{(1-\delta_1)/\kappa} < (1-\varepsilon)\widehat{L}_-(r) \le L_0(r) \le (1+\varepsilon)\widehat{L}_+(r)\right\} \ge 1-\frac{1}{r^{c_{58}}}.$$

Recall that $\mathbb{P}[T_Y(\alpha_\kappa) > 64\log r] \le \frac{2}{r^2}$ for large r. In view of (7.6) and (7.7), we have, for large r,

$$\mathbb{P}\left\{(1-\varepsilon)r^{(1-\delta_1)/\kappa} < (1-\varepsilon)\widehat{L}_-(r) \le L_X^+[H(F(r))] \le (1+\varepsilon)\widehat{L}_+(r)\right\} \ge 1-\frac{1}{r^{c_{59}}}.$$

On the other hand, applying Lemma 7.3 to $z = r^{(1-2\delta_1)/\kappa}$ gives $\mathbb{P}(\sup_{x<0} L_X(H(F(r)),x) > r^{(1-2\delta_1)/\kappa}) \le \mathbb{P}(L_X^{*-}(+\infty) > r^{(1-2\delta_1)/\kappa}) \le c_{47}/r^{(1-2\delta_1)/(\kappa+2)}$ for large r, which implies

$$\mathbb{P}\left\{(1-\varepsilon)\widehat{L}_-(r) \le L_X^*[H(F(r))] \le (1+2\varepsilon)\widehat{L}_+(r)\right\} \ge 1-\frac{1}{r^{c_{59}}}-\frac{c_{47}}{r^{(1-2\delta_1)/(\kappa+2)}},$$

proving the first part of Lemma 3.3. □

Proof of Lemma 3.3: part (ii). In this part, we assume $0 < \kappa \le 1$.

Recall that $H_0(r) = 16\int_0^{\gamma(r)-T_Y(\alpha_\kappa)} \frac{Y(u+T_Y(\alpha_\kappa))}{[1-Y(u+T_Y(\alpha_\kappa))]^2}\mathrm{d}u$, see (7.5). As in Hu et al. ([13], p. 3923), this leads to:

$$H_0(r) = 4\int_0^1 x(1-x)^{\kappa-2}L_\beta[U(\gamma(r)-T_Y(\alpha_\kappa)), S(x)]\mathrm{d}x.$$

By (7.21), we have, for large r,

$$\mathbb{P}\left[I'_-(r) \le H_0(r) \le I'_+(r)\right] \ge 1-\frac{1}{r^{c_{57}}}, \tag{7.22}$$

where

$$\begin{aligned} I'_\pm(r) &:= 4\int_0^1 x(1-x)^{\kappa-2}L_\beta\{\tau_\beta[\lambda t_\pm(r)], S(x)\}\mathrm{d}x \\ &= 4t_\pm(r)\int_0^1 x(1-x)^{\kappa-2}L_{\beta_{t_\pm(r)}}\{\tau_{\beta_{t_\pm(r)}}(\lambda), S(x)/t_\pm(r)\}\mathrm{d}x, \end{aligned}$$

and, as before, $t_\pm(r) = [1 \pm 2(\frac{\lambda}{\kappa})^{\delta_1} r^{-\delta_1}]\frac{\kappa}{\lambda} r$, $\beta_v(s) = \beta(v^2 s)/v$. Let J_β be as defined in (7.12). Then

$$I'_\pm(r) = 4t_\pm(r) J_{\beta_{t_\pm(r)}}[\kappa, t_\pm(r), \lambda].$$

Applying Lemma 7.4 to $d = 1/2$ yields that, for large r,

$$\mathbb{P}\left\{(1-\varepsilon)\widehat{I}_-(r) \le H_0(r) \le (1+\varepsilon)\widehat{I}_+(r)\right\} \ge 1 - \frac{1}{r^{c_{60}}}, \tag{7.23}$$

where $\widehat{I}_\pm(r)$ is defined in (3.12).

In the case $0 < \kappa < 1$, we use once again the inequality $\mathbb{P}[T_Y(\alpha_\kappa) \le 64 \log r] \ge 1 - 2r^{-2}$ (for large r) and the estimate (7.7), to see that $\mathbb{P}[\overline{H}(r) \le c_{61} \log r] \ge 1 - 2r^{-2}$, for some c_{61} and all large r. On the other hand, by Lemma 7.3, $\mathbb{P}[H_-(F(r)) \le \varepsilon r] \ge \mathbb{P}[H_-(+\infty) \le \varepsilon r] \ge 1 - \frac{c_{62}}{r^{(1-\delta_1)\kappa/(\kappa+2)}}$, for all large r. Consequently, by (7.23) and (7.6), for large r,

$$\mathbb{P}\left[(1-\varepsilon)\widehat{I}_-(r) \le H(F(r)) \le (1+\varepsilon)\widehat{I}_+(r) + \frac{4\varepsilon\lambda}{\kappa} t_+(r)\right] \ge 1 - \frac{1}{r^{c_{63}}}.$$

This proves Lemma 3.3 (ii) in the case $0 < \kappa < 1$.

Now we turn to the case $\kappa = 1$. We again have $\mathbb{P}[H_-(F(r)) + \overline{H}(r) \le 2\varepsilon r] \ge 1 - \frac{1}{r^{c_{64}}}$ (for large r). By Biane and Yor [4], $C_{\beta_{t_\pm(r)}} \stackrel{\mathcal{L}}{=} \frac{\pi}{2} C_8^{ca} + c_{65}$, where $c_{65} > 0$. Hence, by (5.10), $\mathbb{P}[C_{\beta_{t_\pm(r)}} > -\pi \log r] \ge 1 - r^{-2}$. Hence, (3.12) gives

$$\mathbb{P}\left\{\widehat{I}_+(r) \ge 16 t_+(r) \log r\right\} \ge 1 - r^{-2}.$$

Consequently, for large r,

$$\mathbb{P}[0 \le H_-(F(r)) + \overline{H}(r) \le \varepsilon \widehat{I}_+(r)] \ge 1 - \frac{1}{r^{c_{66}}},$$

which, in view of (7.23), yields that, for large r,

$$\mathbb{P}[(1-\varepsilon)\widehat{I}_-(r) \le H(F(r)) \le (1+2\varepsilon)\widehat{I}_+(r)] \ge 1 - \frac{1}{r^{c_{67}}}.$$

This proves Lemma 3.3 (ii) in the case $\kappa = 1$. $\square$

The rest of the section is devoted to the proof of Lemmas 7.1–7.4. For the sake of clarity, the proofs of these lemmas are presented in separated subsections.

7.1 Proof of Lemma 7.1

Let $d > 4$ and $R_0 \stackrel{\mathcal{L}}{=} \widetilde{R}_{d-2}(1)$, where $\widetilde{R}$ is a $(d-2)$–dimensional Bessel process. We consider a d–dimensional Bessel process R, starting from R_0. Let $\theta(t) := \int_0^t R^{-2}(s)\mathrm{d}s$. Itô's formula gives $\log R(t) = \log R_0 + M(t) + \frac{d-2}{2}\theta(t)$, where $M(t) := \int_0^t R(s)^{-1}\mathrm{d}\widehat{\beta}(s)$ and $(\widehat{\beta}(t),\ t \ge 0)$

is a Brownian motion. By the Dubins–Schwarz theorem, there exists a Brownian motion $(\widetilde{\beta}(t),\ t \geq 0)$ such that $M(t) = \widetilde{\beta}(\theta(t))$ for all $t \geq 0$. Accordingly,

$$\frac{d-2}{2}\theta(t) = \log R(t) - \log R_0 - \widetilde{\beta}(\theta(t)), \qquad t \geq 0, \tag{7.24}$$

Let $\delta_3 \in (0, \frac{1}{2})$, and let $x = x(t) := \frac{d-2}{6} \frac{1}{(\log t)^{(1/2)-\delta_3}}$.

For large t, we have

$$\mathbb{P}\left(\frac{\log R_0}{\log t} > x\right) = \mathbb{P}\left(R_0 > t^x\right) \leq \exp\left(-(1-\varepsilon)\frac{t^{2x}}{2}\right),$$

and

$$\mathbb{P}\left(\frac{\log R_0}{\log t} < -x\right) = \mathbb{P}\left(R_0 < t^{-x}\right) \leq c_{68}\, t^{-x/(d-2)}.$$

Thus, for all large t,

$$\mathbb{P}\left(\left|\frac{\log R_0}{\log t}\right| > x\right) \leq \exp\left(-(1-\varepsilon)\frac{t^{2x}}{2}\right) + c_{68}\, t^{-x/(d-2)}. \tag{7.25}$$

Let $n := \lceil d \rceil$ be the smallest integer such that $n \geq d$. Since an n-dimensional Bessel process can be realized as the Euclidean modulus of an $\mathbb{R}^n$-valued Brownian motion, it follows from the triangular inequality that

$$R(t) \underset{\mathcal{L}}{\leq} R_0 + \widehat{R}_n(t),$$

where $(\widehat{R}_n(t),\ t \geq 0)$ is an n-dimensional Bessel process starting from 0. Consequently, for large t,

$$\mathbb{P}\left(R(t) > t^{(1/2)+x}\right) \leq \mathbb{P}\left(R_0 > \frac{t^{(1/2)+x}}{2}\right) + \mathbb{P}\left(\widehat{R}_n(t) > \frac{t^{(1/2)+x}}{2}\right) \leq \exp\left(-(1-\varepsilon)\frac{t^{2x}}{4}\right),$$

and

$$\mathbb{P}\left(R(t) < t^{(1/2)-x}\right) \leq c_{68}\, t^{-x/d}.$$

Therefore, for large t,

$$\mathbb{P}\left(\left|\frac{\log R(t)}{\log t} - \frac{1}{2}\right| > x\right) \leq \exp\left(-(1-\varepsilon)\frac{t^{2x}}{4}\right) + c_{68}\, t^{-x/d}. \tag{7.26}$$

Define

$$\begin{aligned}
E_{12} &:= \left\{\left|\frac{\log R(t)}{\log t} - \frac{1}{2}\right| \leq x\right\} \cap \left\{\left|\frac{\log R_0}{\log t}\right| \leq x\right\} \\
E_{13} &:= \left\{\frac{d-2}{2}\theta(t) < \log t\right\}. \\
E_{14} &:= \left\{\sup_{0 \leq s \leq 2(\log t)/(d-2)} |\widetilde{\beta}(s)| \leq x \log t\right\}.
\end{aligned}$$

By (7.25) and (7.26), we have, for large t,

$$\mathbb{P}\left(E_{12}^c\right) \le 2\exp\left(-(1-\varepsilon)\frac{t^x}{4}\right) + c_{69}\, t^{-x/d}.$$

We now estimate $\mathbb{P}(E_{12} \cap E_{13}^c)$. We first observe that on E_{12}, we have, by (7.24),

$$\left|\widetilde{\beta}(\theta(t)) + \frac{d-2}{2}\theta(t) - \frac{1}{2}\log t\right| \le 2x \log t.$$

We claim that $E_{12} \cap E_{13}^c \subset \{|\widetilde{\beta}(\theta(t))| > \frac{d-2}{6}\theta(t)\}$ for large t. Indeed, on the event $E_{12} \cap E_{13}^c \cap \{|\widetilde{\beta}(\theta(t))| \le \frac{d-2}{6}\theta(t)\}$,

$$\frac{d-2}{2}\theta(t) \le \left(\frac{1}{2} + 2x\right)\log t - \widetilde{\beta}(\theta(t)) \le \left(\frac{1}{2} + 2x\right)\log t + \frac{d-2}{6}\theta(t),$$

which implies $\frac{d-2}{2}\theta(t) \le (\frac{3}{4} + 3x)\log t$, which, for large t, contradicts $\frac{d-2}{2}\theta(t) > \log t$ on E_{13}^c. Therefore, $E_{12} \cap E_{13}^c \subset \{|\widetilde{\beta}(\theta(t))| > \frac{d-2}{6}\theta(t)\}$ holds for all large t, from which it follows that

$$\begin{aligned}
\mathbb{P}(E_{12} \cap E_{13}^c) &\le \mathbb{P}\left(\sup_{s \ge 2(\log t)/(d-2)} \frac{|\widetilde{\beta}(s)|}{s} > \frac{d-2}{6}\right) \\
&= \mathbb{P}\left(\sup_{u \ge 1} \frac{|\widetilde{\beta}(u)|}{u} > \frac{d-2}{6}\sqrt{2(\log t)/(d-2)}\right) \\
&= \mathbb{P}\left(\sup_{0 \le v \le 1} |\widetilde{\beta}(v)| > \frac{d-2}{6}\sqrt{2(\log t)/(d-2)}\right) \\
&\le \exp\left(-(1-\varepsilon)\frac{(d-2)\log t}{36}\right).
\end{aligned}$$

Since $\mathbb{P}(E_{14}^c) \le \exp[-(1-\varepsilon)\frac{d-2}{4}x^2 \log t]$ (for large t), we have, for large t,

$$\begin{aligned}
\mathbb{P}(E_{12}^c \cup E_{13}^c \cup E_{14}^c) &\le \mathbb{P}(E_{12}^c) + \mathbb{P}(E_{12} \cap E_{13}^c) + P(E_{12} \cap E_{13} \cap E_{14}^c) \\
&\le 2\exp\left(-(1-\varepsilon)\frac{t^x}{4}\right) + c_{69}\, t^{-x/d} \\
&\quad + \exp\left(-(1-\varepsilon)\frac{(d-2)\log t}{36}\right) + \exp\left(-(1-\varepsilon)\frac{d-2}{4}x^2 \log t\right),
\end{aligned}$$

which is smaller than $\exp(-c_{70}\, x^2 \log t)$. This completes the proof of Lemma 7.1 via the trivial observation that $E_{12} \cap E_{13} \cap E_{14} \subset \{|\frac{\theta(t)}{\log t} - \frac{1}{d-2}| \le \frac{6x}{d-2}\}$. $\square$

7.2 Proof of Lemma 7.2

Let $v > 0$. Recall that for $x > 0$, $\beta_v(x) := (1/v)\beta(v^2 x)$, and notice that $v^2\tau_{\beta_v}(x) = \tau_\beta(xv)$ almost surely. Then,

$$E_{10} = \left\{\tau_{\beta_v}[(1-v^{-\delta_1})\lambda] \le \frac{U(v)}{v^2} \le \tau_{\beta_v}[(1+v^{-\delta_1})\lambda]\right\}. \tag{7.27}$$

For $\delta_1 > 0$, define

$$\begin{aligned}
\varepsilon_1 = \varepsilon_1(v,s) &:= \frac{1}{4}\int_0^1 (1-x)^\kappa \left[L_{\beta_v}(s, \frac{S(x)}{v}) - L_{\beta_v}(s,0)\right] \mathrm{d}x, \qquad s \ge 0,\\
E_{15} &:= \left\{\sup_{0\le s\le \tau_{\beta_v}(2\lambda)} |\varepsilon_1(v,s)| < v^{-\delta_1}\right\}.
\end{aligned}$$

Recall (see Hu et al. [13] p. 3924) that $E_{15} \subset E_{10}$. Thus it remains to prove that for δ_1 small enough, $\mathbb{P}(E_{15}^c) \le 1/v^{1/4-5\delta_1}$ for large u. Notice that for $s \ge 0$,

$$\begin{aligned}
|\varepsilon_1| &\le \left(\int_{\{S(x)>\sqrt{v}\}} + \int_{\{S(x)<-\sqrt{v}\}} + \int_{\{|S(x)|\le\sqrt{v}\}}\right) \frac{(1-x)^\kappa}{4}\left|L_{\beta_v}(s, \frac{S(x)}{v}) - L_{\beta_v}(s,0)\right| \mathrm{d}x\\
&=: \varepsilon_2(v,s) + \varepsilon_3(v,s) + \varepsilon_4(v,s)
\end{aligned} \tag{7.28}$$

with obvious notation.

By (7.15), we have, for all large v (and all $s \ge 0$)

$$\varepsilon_2(v,s) \le \frac{1}{4}\int_{1-\left(\frac{2}{\kappa\sqrt{v}}\right)^{1/\kappa}}^1 (1-x)^\kappa \sup_{u\ge 0}\left[L_{\beta_v}(s,u) + L_{\beta_v}(s,0)\right] \mathrm{d}x,$$

which implies

$$\begin{aligned}
\sup_{0\le s\le \tau_{\beta_v}(2\lambda)} \varepsilon_2(v,s) &\le \left(\frac{2}{\kappa\sqrt{v}}\right)^{\frac{1}{\kappa}+1} \sup_{0\le s\le\tau_{\beta_v}(2\lambda)}\sup_{u\ge 0}\left[L_{\beta_v}(s,u) + L_{\beta_v}(s,0)\right]\\
&= \left(\frac{2}{\kappa\sqrt{v}}\right)^{\frac{1}{\kappa}+1} \sup_{u\ge 0}\left[L_{\beta_v}(\tau_{\beta_v}(2\lambda),u) + 2\lambda\right].
\end{aligned}$$

By the second Ray–Knight theorem (Fact 2.2), $Z := (L_{\beta_v}(\tau_{\beta_v}(2\lambda),u),\ u \ge 0)$ is a 0–dimensional squared Bessel process starting from 2λ. Hence, for large v,

$$\mathbb{P}\left(\sup_{0\le s\le\tau_{\beta_v}(2\lambda)} \varepsilon_2(v,s) \ge \left(\frac{2}{\kappa\sqrt{v}}\right)^{\frac{1}{\kappa}+1}(\sqrt{v}+2\lambda)\right) \le \mathbb{P}\left(\sup_{u\ge 0} Z(u) \ge \sqrt{v}\right) = \frac{2\lambda}{\sqrt{v}}. \tag{7.29}$$

Similarly (this time, using $S(x) \sim \log x$, $x \to 0$), we have, for large v,

$$\sup_{0\le s\le\tau_{\beta_v}(2\lambda)} \varepsilon_3(v,s) \le \exp\left(-\frac{\sqrt{v}}{2}\right)\sup_{u\le 0}\left[L_{\beta_v}(\tau_{\beta_v}(2\lambda),u) + 2\lambda\right], \tag{7.30}$$

which leads to

$$\mathbb{P}\left(\sup_{0\le s\le \tau_{\beta_v}(2\lambda)} \varepsilon_3(v,s) \ge \exp\left(-\frac{\sqrt{v}}{2}\right)(\sqrt{v}+2\lambda)\right) \le \frac{2\lambda}{\sqrt{v}}. \tag{7.31}$$

To estimate $\varepsilon_4(v,s)$, we note that

$$\varepsilon_4(v,s) \le \sup_{|u|\le 1/\sqrt{v}} |L_{\beta_v}(s,u) - L_{\beta_v}(s,0)|. \tag{7.32}$$

Let $\varepsilon \in (0,1/2)$, $t_v > 0$, $\gamma \ge 1$ and define $(\beta_v)^*_{t_v} := \sup_{0\le s\le t_v} |\beta_v(s)|$. Applying Barlow and Yor ([2], (ii)) to the continuous martingale $\beta_v(.\wedge t_v)$, we see that for some constant $C_{\gamma,\varepsilon} > 0$,

$$\left\|\sup_{0\le s\le t_v, a\ne b} \frac{|L_{\beta_v}(s,b) - L_{\beta_v}(s,a)|}{|b-a|^{1/2-\varepsilon}}\right\|_\gamma \le C_{\gamma,\varepsilon}\left\|[(\beta_v)^*_{t_v}]^{1/2+\varepsilon}\right\|_\gamma.$$

Then, by Chebyshev's inequality, for $\alpha > 0$,

$$\mathbb{P}\left(\sup_{0\le s\le t_v,\, a\ne b} \frac{|L_{\beta_v}(s,b) - L_{\beta_v}(s,a)|}{|b-a|^{1/2-\varepsilon}} \ge \alpha\right) \le \frac{(\sqrt{t_v})^{(1/2+\varepsilon)\gamma}}{\alpha^\gamma}\left[C_{\gamma,\varepsilon}\left\|[(\beta_v)^*_1]^{1/2+\varepsilon}\right\|_\gamma\right]^\gamma. \tag{7.33}$$

Let

$$E_{16} := \left\{\sup_{0\le s\le \tau_{\beta_v}(2\lambda), a\ne b} \frac{|L_{\beta_v}(s,b) - L_{\beta_v}(s,a)|}{|b-a|^{1/2-\varepsilon}} \le v^{\frac{1}{2}\left(\frac{1}{2}-2\varepsilon\right)}\right\}.$$

Recall (7.32). On E_{16}, we have

$$\sup_{0\le s\le \tau_{\beta_v}(2\lambda)} \varepsilon_4(v,s) \;\le\; \left(\frac{1}{\sqrt{v}}\right)^{\frac{1}{2}-\varepsilon} v^{\frac{1}{2}\left(\frac{1}{2}-2\varepsilon\right)} = v^{-\varepsilon/2}. \tag{7.34}$$

We choose $\gamma := 2$ and $t_v := v^{\frac{1/4-\varepsilon}{1/2+\varepsilon}}$ to see that

$$\begin{aligned}
\mathbb{P}(E_{16}(v)^c) &\le \mathbb{P}(\tau_{\beta_v}(2\lambda) > t_v) + \mathbb{P}\left(\sup_{0\le s\le t_v, a\ne b} \frac{|L_{\beta_v}(s,b) - L_{\beta_v}(s,a)|}{|b-a|^{1/2-\varepsilon}} \ge v^{\frac{1}{2}\left(\frac{1}{2}-2\varepsilon\right)}\right)\\
&\le \frac{4\lambda}{\sqrt{2\pi}} v^{\frac{\varepsilon-1/4}{1+2\varepsilon}} + c_{71} v^{-1/4+\varepsilon}\\
&\le v^{-1/4+2\varepsilon}/2
\end{aligned}$$

for all large v (if ε is small enough). Combining this with (7.28), (7.29), (7.31) and (7.34), we obtain that, for $\varepsilon > 0$ small enough,

$$\mathbb{P}\left(\sup_{0\le s\le \tau_{\beta_v}(2\lambda)} |\varepsilon_1(v,s)| \ge 2v^{-\varepsilon/2}\right) \le v^{-1/4+2\varepsilon}.$$

This gives, with the choice of $\delta_1 := 2\varepsilon/5$, $\mathbb{P}(E^c_{15}) \le v^{-1/4+5\delta_1}$ (for large v). $\square$

7.3 Proof of Lemma 7.3

We notice that

$$\begin{aligned} L_X^{*-}(+\infty) &= \sup_{r\geq 0}\sup_{x<0}\left\{e^{-W_\kappa(x)}L_B[\sigma_B(A(r)),A(x)]\right\} \\ &\leq \left(\sup_{x<0} e^{-W_\kappa(x)}\right)\left(\sup_{y<0} L_B[\sigma_B(A_\infty),y]\right). \end{aligned}$$

For $a>0$, $\alpha>0$ and $b>0$, let

$$\begin{aligned} E_{17} &:= \left\{\sup_{x<0} e^{-W_\kappa(x)} \leq a\right\}, \\ E_{18} &:= \{A_\infty \leq \alpha\}, \\ E_{19} &:= \left\{\sup_{y<0} L_B[\sigma_B(\alpha),y] \leq b\right\}. \end{aligned}$$

By the first Ray–Knight theorem (Fact 2.1), there exist two Bessel processes R_2 and R_0, of dimensions 2 and 0 respectively, starting from 0 and $R_2(\alpha)$, such that $L_B(\sigma_B(\alpha),x)$ is equal to $R_2^2(\alpha-x)$ for $x\in[0,\alpha]$ and to $R_0^2(-x)$ for $x<0$. Hence, for $\alpha\leq b$,

$$\begin{aligned} \mathbb{P}(E_{19}^c) &= \mathbb{P}(R_2^2(\alpha)>b)+\int_0^b \mathbb{P}_x\left(\sup_{y>0} R_0^2(y)>b\right)\mathbb{P}(R_2^2(\alpha)\in[x,x+\mathrm{d}x]) \\ &\leq 4\exp\left(-\frac{b}{4\alpha}\right)+\mathbb{E}\left(\frac{R_2^2(\alpha)}{b}\mathbf{1}_{\{R_2^2(\alpha)\leq b\}}\right) \\ &\leq \left[16+\mathbb{E}\left(R_2^2(1)\right)\right]\frac{\alpha}{b} =: c_{72}\frac{\alpha}{b}. \end{aligned}$$

Now, let $a:=z^{\frac{1}{\kappa+2}}$, $\alpha:=z^{\frac{1}{\kappa+2}}$ and $b:=z^{\frac{\kappa+1}{\kappa+2}}$. Notice that $L_X^{*-}(+\infty)\leq z$ on $E_{17}\cap E_{18}\cap E_{19}$, and recall $A_\infty \stackrel{\mathcal{L}}{=} 2/\gamma_\kappa$ (see Dufresnes, [9]), where γ_κ is a gamma variable of parameter κ. We have for z large enough,

$$\begin{aligned} \mathbb{P}(L_X^{*-}(+\infty)>z) &\leq \mathbb{P}(E_{17}^c)+\mathbb{P}(E_{18}^c)+\mathbb{P}(E_{19}^c) \\ &\leq \frac{1}{a^\kappa}+\frac{1}{\kappa\Gamma(\kappa)}\left(\frac{2}{\alpha}\right)^\kappa+c_{72}\frac{\alpha}{b} \\ &\leq c_{73}z^{-\frac{\kappa}{\kappa+2}}. \end{aligned} \tag{7.35}$$

This gives (7.10) for z large enough.

It remains to prove (7.11). Define for $c>0$,

$$\begin{aligned} E_{20} &:= \left\{\min_{0\leq s\leq\sigma_B(A_\infty)} B(s) > -A_\infty z^{\frac{\kappa+1}{\kappa+2}}\right\}, \\ E_{21} &:= \{|A^{-1}(-z)|\leq c\log z\}. \end{aligned}$$

On $E_{17} \cap E_{18} \cap E_{19} \cap E_{20} \cap E_{21}$, we have for $r \geq 0$,

$$\begin{aligned} H_-(+\infty) &= \lim_{r\to+\infty} \int_{\min_{0\leq s\leq\sigma_B(A(r))} B(s)}^{0} \exp\{-2W_\kappa[A^{-1}(x)]\} L_B\{\sigma_B[A(r)], x\}\mathrm{d}x \\ &= \lim_{r\to+\infty} \int_{A^{-1}\left(\min_{0\leq s\leq\sigma_B(A(r))} B(s)\right)}^{0} L_X(H(r), x)\mathrm{d}x \\ &\leq \left| A^{-1}\left(\min_{0\leq s\leq\sigma_B(A_\infty)} B(s) \right) \right| L_X^{*-}(+\infty) \\ &\leq |A^{-1}(-z)| L_X^{*-}(+\infty) \\ &\leq cz\log z. \end{aligned} \tag{7.36}$$

As B is independent of A_∞, we have

$$\mathbb{P}\left(E_{20}^c | A_\infty\right) = \frac{A_\infty}{A_\infty + A_\infty z^{\frac{\kappa+1}{\kappa+2}}} \leq \frac{1}{z^{\frac{\kappa+1}{\kappa+2}}}. \tag{7.37}$$

Moreover, for $c > 2/\kappa$, and $\varepsilon > 0$,

$$\begin{aligned} \mathbb{P}(E_{21}^c) &= \mathbb{P}(-z < A(-c\log z)) \\ &= \mathbb{P}\left(z > \int_0^{c\log z} \exp(W(-u) + \kappa u/2)\mathrm{d}u \right) \\ &\leq \mathbb{P}\left[z > \exp\left(\inf_{0\leq u\leq c\log z} W(-u) \right) \frac{2}{\kappa}(z^{\kappa c/2} - 1) \right] \\ &\leq \mathbb{P}\left(\inf_{0\leq u\leq c\log z} W(u) < (1 - \kappa c/2 + \varepsilon)\log z \right) \\ &\leq 2z^{-\frac{1}{2c}\left(\frac{\kappa c}{2} - 1 - \varepsilon\right)^2} \end{aligned}$$

for all large z. Choosing c large enough, this, together with (7.35), (7.36) and (7.37), gives (7.11). □

7.4 Proof of Lemma 7.4

Assume $0 < \kappa \leq 1$. Let $0 < d < 1$, and consider a Brownian motion β. Let $\varepsilon > 0$ be a small constant. Recall $S(y) = \int_{\alpha_\kappa}^{y} \frac{\mathrm{d}x}{x(1-x)^{1+\kappa}}$ and notice that

$$1 - S^{-1}(u) \underset{u\to+\infty}{\sim} (\kappa u)^{-1/\kappa}.$$

Therefore, there exists $x_\varepsilon > 0$ such that

$$\forall u \geq x_\varepsilon, \qquad \frac{[1 - S^{-1}(u)]^{2\kappa-1}}{(\kappa u)^{1/\kappa-2}} \in (1-\varepsilon, 1+\varepsilon) \quad \text{and} \quad S^{-1}(u) \geq (1-\varepsilon).$$

Let $g(t) := t^{\varepsilon-1}$, and write

$$\begin{aligned} J_\beta(\kappa,t,\lambda) &= \int_0^1 y(1-y)^{\kappa-2} L_\beta(\tau_\beta(\lambda), S(y)/t)\mathrm{d}y \\ &= \left(\int_{\{S(y)/t\le -g(t)\}} + \int_{\{-g(t)<S(y)/t\le 0\}} + \int_{\{0<S(y)\le x_\varepsilon\}} + \int_{\{x_\varepsilon<S(y)\}}\right)\cdots \mathrm{d}y \\ &:= J_1+J_2+J_3+J_4, \end{aligned}$$

with obvious notation.

As in (7.30), for large t,

$$J_1 \le \exp\left(-\frac{tg(t)}{2}\right)\left(\sup_{s\ge 0} Z(s)\right),$$

where Z is a 0–dimensional squared Bessel process starting from λ (by the second Ray–Knight theorem stated in Fact 2.2). Hence,

$$\mathbb{P}\left[J_1 \ge \exp\left(-\frac{t^\varepsilon}{2}\right)t^d\right] \le \frac{\lambda}{t^d}. \tag{7.38}$$

Fix a large constant $\gamma>0$, and define

$$\begin{aligned} E_{22} &:= \{\tau_\beta(\lambda)\le t^{2d}\}, \\ E_{23} &:= \left\{\sup_{0\le s\le t^{2d},\, a\ne b} \frac{|L_\beta(s,b)-L_\beta(s,a)|}{|b-a|^{1/2-\varepsilon}} \le t^{d(1/2+\varepsilon+1/\gamma)}\right\}. \end{aligned}$$

Recall that $S(\alpha_\kappa)=0$. On the event $E_{22}\cap E_{23}$ and for all large t,

$$\begin{aligned} \kappa^{2-1/\kappa}t^{1-1/\kappa}J_3 &\le \left[\kappa^{2-1/\kappa}\int_{\alpha_k}^{S^{-1}(x_\varepsilon)} y(1-y)^{\kappa-2}\mathrm{d}y\right] t^{1-1/\kappa} \sup_{0\le x\le x_\varepsilon/t} L_\beta(\tau_\beta(\lambda),x) \\ &\le c_{74}t^{1-1/\kappa}\left[\lambda + t^{d(1/2+\varepsilon+1/\gamma)}(x_\varepsilon/t)^{\frac12-\varepsilon}\right] \\ &\le 2\lambda c_{74}t^{1-1/\kappa}. \end{aligned}$$

Since $\mathbb{P}(E_{22}^c)\le c_{75}/t^d$ and $\mathbb{P}(E_{23}^c)\le c_{76}/t^d$ (see (7.33)), we obtain, for large t,

$$\mathbb{P}\left(J_3 \le c_{77}\right) \ge 1-\frac{c_{78}}{t^d}. \tag{7.39}$$

To estimate J_2, we note that, on $E_{22}\cap E_{23}$, for all large t,

$$\begin{aligned} J_2 &\le \left[\int_0^{\alpha_k} y(1-y)^{\kappa-2}\mathrm{d}y\right] \sup_{-g(t)\le s\le 0} L_\beta(\tau_\beta(\lambda),s) \\ &\le \alpha_\kappa(1-\alpha_k)^{\kappa-2}\left[\lambda + t^{d(1/2+\varepsilon+1/\gamma)}(t^{\varepsilon-1})^{\frac12-\varepsilon}\right] \\ &\le 2\alpha_\kappa(1-\alpha_k)^{\kappa-2}\lambda. \end{aligned}$$

Therefore, for large t,

$$\mathbb{P}\left(J_2 \leq c_{79}\right) \geq 1 - \frac{c_{80}}{t^d}. \tag{7.40}$$

To estimate J_4, we observe that

$$J_4 = \kappa^{1/\kappa-2}\ t^{1/\kappa-1} \int_{x_\varepsilon/t}^{+\infty} \big(S^{-1}(tx)\big)^2 \frac{\big(1-S^{-1}(tx)\big)^{2\kappa-1}}{(\kappa t)^{1/\kappa-2}} L_\beta(\tau_\beta(\lambda),x)\mathrm{d}x.$$

Therefore

$$\begin{aligned}(1-\varepsilon)^3\int_{x_\varepsilon/t}^{+\infty} x^{1/\kappa-2}L_\beta(\tau_\beta(\lambda),x)\mathrm{d}x &\leq \kappa^{2-1/\kappa}t^{1-1/\kappa}J_4 \\ &\leq (1+\varepsilon)\int_{x_\varepsilon/t}^{+\infty} x^{1/\kappa-2}L_\beta(\tau_\beta(\lambda),x)\mathrm{d}x. \end{aligned}\tag{7.41}$$

Proof of Lemma 7.4: part (i). We assume $0<\kappa<1$.

On $E_{22}\cap E_{23}$, for large t, we have $\int_0^{x_\varepsilon/t} x^{1/\kappa-2}L_\beta(\tau_\beta(\lambda),x)\mathrm{d}x \leq c_{81}t^{1-1/\kappa}$. It follows from (7.41) that, for large t,

$$\mathbb{P}\left[(1-\varepsilon)^3K_\beta(\kappa)-(1-\varepsilon)^3c_{81}t^{1-1/\kappa} \leq \kappa^{2-1/\kappa}t^{1-1/\kappa}J_4 \leq (1+\varepsilon)K_\beta(\kappa)\right] \geq 1-\frac{c_{82}}{t^d},$$

where $K_\beta(\kappa)$ is defined in (3.7).

Since $J_\beta(\kappa,t,\lambda) = J_1+J_2+J_3+J_4$, the above estimate, combined with (7.38), (7.39) and (7.40), yields that, for large t,

$$\mathbb{P}\left\{(1-\varepsilon)^3K_\beta(\kappa)-c_{50}t^{1-1/\kappa} \leq \kappa^{2-1/\kappa}t^{1-1/\kappa}J_\beta(\kappa,t,\lambda) \leq (1+\varepsilon)K_\beta(\kappa)+c_{50}t^{1-1/\kappa}\right\}$$

is greater than $1-\frac{c_{83}}{t^d}$. □

Proof of Lemma 7.4: part (ii). We assume $\kappa=1$.

By the definition of C_β (see (3.8)), we have

$$\begin{aligned}\int_{x_\varepsilon/t}^{\infty}\frac{L_\beta(\tau_\beta(8),x)}{x}\mathrm{d}x &= C_\beta - \int_0^1 \frac{L_\beta(\tau_\beta(8),x)-8}{x}\mathrm{d}x + \int_{x_\varepsilon/t}^1 \frac{L_\beta(\tau_\beta(8),x)}{x}\mathrm{d}x \\ &= C_\beta - \int_0^{x_\varepsilon/t}\frac{L_\beta(\tau_\beta(8),x)-8}{x}\mathrm{d}x + 8\log t - 8\log x_\varepsilon.\end{aligned}$$

On $E_{22}\cap E_{23}$, for large t,

$$\int_0^{x_e/t}\frac{|L_\beta(\tau_\beta(8),x)-8|}{x}\mathrm{d}x \leq \int_0^{x_e/t}\frac{t^{d(1/2+\varepsilon+1/\gamma)}x^{1/2-\varepsilon}}{x}\mathrm{d}x \leq \frac{t^{d(1/2+\varepsilon+1/\gamma)}x_\varepsilon^{1/2-\varepsilon}}{(1/2-\varepsilon)t^{1/2-\varepsilon}} \leq \varepsilon.$$

As in (5.10), $\mathbb{P}(C_\beta + 8\log t < \log t) \le r^{-7}$. Therefore, by means of (7.41), we have, for large t,

$$\mathbb{P}\left\{(1-\varepsilon)^4[C_\beta + 8\log t] \le J_4 \le (1+\varepsilon)^2[C_\beta + 8\log t]\right\} \ge 1 - \frac{c_{84}}{t^d}.$$

Since $J_\beta(1,t,8) = J_1 + J_2 + J_3 + J_4$ (in the case $\kappa = 1$), this inequality, together with (7.38), (7.39) and (7.40), yields that, for large t,

$$\mathbb{P}\left\{(1-\varepsilon)^4[C_\beta + 8\log t] \le J_\beta(1,t,8) \le (1+\varepsilon)^3[C_\beta + 8\log t]\right\} \ge 1 - \frac{c_{85}}{t^d},$$

as desired. $\square$

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